Self-Organizing Synchronization with Inhibitory-Coupled Oscillators: Convergence and Robustness

JOHANNES KLINGLMAYR, University of Klagenfurt
CHRISTIAN BETTSTETTER, University of Klagenfurt and Lakeside Labs

Solutions for time synchronization based on coupled oscillators operate in a self-organizing and adaptive manner and can be applied to various types of dynamic networks. The basic idea was inspired by swarms of fireflies, whose flashing dynamics shows an emergent behavior. This article introduces such a synchronization technique whose main components are "inhibitory coupling" and "self-adjustment." Based on this new technique, a number of contributions are made. First, we prove that inhibitory coupling can lead to perfect synchrony independent of initial conditions for delay-free environments and homogeneous oscillators. Second, relaxing the assumptions to systems with delays and different phase rates, we prove that such systems synchronize up to a certain precision bound. We derive this bound assuming inhomogeneous delays and show by simulations that it gives a good estimate in strongly-coupled systems. Third, we show that inhibitory coupling with self-adjustment quickly leads to synchrony with a precision comparable to that of excitatory coupling. Fourth, we analyze the robustness against faulty members performing incorrect coupling. While the specific precision loss encountered by such disturbances depends on system parameters, the system always regains synchrony for the investigated scenarios.

Categories and Subject Descriptors: C.2 [**Computer-Communication Networks**]: Distributed Systems; C.4 [**Performance of Systems**]: Fault Tolerance; F.2.m [**Analysis of Algorithms and Problem Complexity**]: Miscellaneous; G.m [**Mathematics of Computing**]: Miscellaneous

General Terms: Algorithms, Performance, Reliability, Theory

Additional Key Words and Phrases: Synchronization, self-organization, engineering emergent behavior, networked systems, pulse coupled oscillators, inhibitory coupling, heterogeneous system, delayed system.

ACM Reference Format:
Klinglmayr, J., Bettstetter, C. 201?. Self-organizing synchronization with inhibitory-coupled oscillators: convergence and robustness. ACM Trans. Autonom. Adapt. Syst. 6, ?, Article A (March 2011), 22 pages.
DOI = 10.1145/0000000.0000000 http://doi.acm.org/10.1145/0000000.0000000

1. INTRODUCTION

The synchronous flashing of fireflies and other synchronization phenomena are prime examples for emergent and adaptive behavior in natural systems [Strogatz 2003]. Early studies on firefly synchronization in the first half of the 20th century were based on the hypothesis that a particular leading firefly controls the system or that synchrony is triggered by external events [Camazine et al. 2001]. It took some time until it was generally accepted that

---

This work has been partly performed in the research cluster Lakeside Labs. It is supported by the Austrian Research Promotion Agency (FFG) within the FIT-IT program under grant 825893 (ROSSY); and by the European Regional Development Fund, the Carinthian Economic Promotion Fund (KWF), and the state of Austria under grant 20214/17094/24770 (Triple-S).
Authors' addresses: J. Klinglmayr and C. Bettstetter, Institute of Networked and Embedded Systems (NES), University of Klagenfurt, Lakeside B02, 9020 Klagenfurt, Austria; C. Bettstetter, Lakeside Labs GmbH, Lakeside B04, 9020 Klagenfurt, Austria.

DOI 10.1145/0000000.0000000 http://doi.acm.org/10.1145/0000000.0000000

this phenomenon arises from a completely distributed mechanism in which fireflies mutually interact and influence each other. Scientific experiments showed that every firefly has its own flashing rhythm when left alone, and it adjusts this flashing rhythm when receiving a flash from another firefly [Buck et al. 1981]. Another step toward a better understanding of firefly synchronization was made by Winfree [1967], who presented an analysis on the behavior of coupled oscillators in biological systems and stated that "individual rythmicity with phase-dependent sensitivity to mutual influences can give rise to ... striking community synchronization." It became more and more clear that synchronization may arise from within the system. It is based on simple and local interactions between the individual fireflies.

Following these results, it was suggested to model fireflies as a system of pulse coupled oscillators. Each individual firefly is an oscillator running at an internal rate. Its state is represented by a phase $\phi(t)$, which evolves over time t from 0 to 1, resets back to 0, evolves again, and so on. The oscillators interact with each other by sending and receiving pulse signals. Whenever $\phi = 1$, an oscillator emits a pulse signal (fire event). Upon reception of such a signal, each oscillator immediately adjusts its own phase by performing a phase jump $\Delta\phi(t)$. If each member in a group of oscillators follows such a simple rule, synchronization will eventually emerge spontaneously. A rigorous mathematical proof that such a self-organizing, multi-agent system converges to synchrony was presented by Mirollo and Strogatz [1990]. This breakthrough gave rise to further theoretical analysis of pulse coupled oscillators (see, e.g., [Ermentrout 1991; Ernst et al. 1995; Abbott and van Vreeswijk 1993; Timme 2002]) and their application in various fields of science. To give some examples, coupled oscillators can be used to explain the synchronous firing of neurons, the formation of earthquakes, forest fires, mass extinctions, sleep cycles, and bridge vibrations.

More recently, the theory of firefly synchronization also found its application in research on computer and communication networks. A one-to-one transfer of the Mirollo-Strogatz approach is, however, infeasible, as the inherent characteristics of most technologies are not compatible with the modeling assumptions there. Thus, modifications and extensions have been proposed in the literature. These proposals take into account propagation and processing delays [Mathar and Mattfeldt 1996], channel attenuation and noise [Hong and Scaglione 2005], general network topologies without all-to-all coupling [Lucarelli and Wang 2004; Tyrrell et al. 2010], and the fact that infinitely short pulses are infeasible in most communication technologies [Werner-Allen et al. 2005; Tyrrell et al. 2010]. Synchronization algorithms inspired by fireflies have been developed for a variety of network types, such as sensor networks [Werner-Allen et al. 2005], overlay networks [Babaoglu et al. 2007], and cellular communication networks [Tyrrell and Auer 2008]. Case studies using firefly synchronization on real hardware are presented in [Leidenfrost and Elmenreich 2009] and [Pagliari and Scaglione 2011].

Despite these advances in biologically-inspired synchronization for technical systems, an important issue toward practical deployment did not receive much attention so far: the robustness of the synchronization algorithm against *faulty* devices. So far, research on firefly-inspired synchronization in technical systems broadly assumes that all members in the network strictly follow the rules of the synchronization algorithm. What happens, however, if one or more members misbehave in some manner? This question is important for two scenarios: First, devices may always become defective and work in an unintended manner. Second, intruding malicious members might want to disturb the network operation. As long as the behavior of self-organized synchronization to such events remains unexamined, self-organized synchronization will have little chance to be deployed in practice. First steps to analyze the effects of faulty devices were made by Tyrrell et al. [2010] and Leidenfrost et al. [2010]. Other types of errors, such as message loss and missed signal detection, are investigated e.g. in [Hong and Scaglione 2005; Babaoglu et al. 2007; Tyrrell et al. 2010].

To approach the problem of robustness in more detail, in particular to propose more robust synchronization models, the following observation might be important: Almost all approaches for biologically-inspired synchronization in computer and communication systems assume that an oscillator performs a *positive* phase jump $\Delta\phi(t) \geq 0$ whenever receiving a pulse ("excitatory coupling"). In contrast to this, neuroscience publications by van Vreeswijk et al. [1994] and Ernst et al. [1995] suggest to employ *negative* phase jumps $\Delta\phi(t) \leq 0$ ("inhibitory coupling") and conclude that negative jumps can have certain constructive effects on the synchronization process. Such negative coupling was analyzed in more detail by Timme et al. [2002], who analyze the collective network dynamics for arbitrary network topologies and for initial phases close to synchrony. Conclusions for arbitrary initial conditions could, however, not be drawn.

Two major questions arise: Can we show that inhibitory coupling leads to synchronization of all oscillators independent of the initial conditions? Can inhibitory coupling improve the robustness against faulty behavior compared to excitatory coupling? The first question will be answered positively in this article. The key idea is that a firing oscillator omits the phase reset upon firing but just performs a negative phase jump due to self-coupling. The second question will be answered in an ambiguous manner: While inhibitory coupling is indeed able to improve the robustness significantly for well-chosen system parameters, it may also react in a less robust manner for other system parameters.

The main contribution of this article is a solution for self-organizing network synchronization that (a) is proven to lead to synchrony of all nodes independent of the initial conditions, (b) works in networks with and without delays, (c) works with heterogeneous systems of oscillators, and (d) can have advantageous effects on the robustness against certain faulty behavior compared to previous approaches in strongly-coupled systems.

This article is structured as follows: Section 2 gives a brief introduction on the theory of pulse coupled oscillators. It explains the basic synchronization scheme, motivates the introduction of a refractory state needed for systems with delays, and compares inhibitory with excitatory coupling. This section also covers related work. Section 3 proposes a new synchronization scheme based on inhibitory-coupled oscillators and proves its convergence by analyzing the synchronization precision over time. The convergence bound is even valid for systems with variable delays and heterogeneous oscillator rates. Finally, Section 4 investigates the synchronization performance of this scheme and its robustness with respect to erroneous fire events and failure of fire event detection. It is shown that, if the coupling is high, the proposed inhibitory-coupled synchronization scheme can be more robust to such faults than existing excitatory-coupled synchronization schemes. Some preliminary results of this article have been published in [Klinglmayr et al. 2009]. For convenience of notation, time variables and time values are considered to have unit one in this article.

2. BACKGROUND AND RELATED WORK ON PULSE COUPLED OSCILLATORS

2.1. Basic Formulation

Different approaches have been taken to describe synchronization phenomena in systems of coupled oscillators. Winfree [1967] and Kuramoto [1975] use models where the interactions between the oscillators are smooth. Such a time-continuous coupling model has some restrictions in its analysis as it is difficult to unveil the system dynamics and convergence. In contrast to this, Mirollo and Strogatz [1990] use a model where the interactions are episodic and pulse-like. Using such a non-linear operator based approach, the system dynamics becomes more visible and convergence conclusions can be stated. In the following we describe a variant of the frequently-used Mirollo-Strogatz model for pulse coupled oscillators.

An oscillator i is represented by its *phase* $\phi_i(t)$, indicating its position on its oscillation cycle. It starts at $\phi_i(t) = 0$, evolves over time t until it reaches $\phi_i(t) = 1$, then resets to $\phi_i(t) = 0$ and starts again. The time period it takes from phase 0 to 1 is the *cycle* of an oscillator. The cycle length is $\omega = 1$. The *phase rate* $\dot{\phi}_i = \frac{d\phi_i}{dt}$ is assumed to be constant over time and equal for all oscillators. Thus, each oscillator is defined by the phase rate

$\dot{\phi}_i = 1$. Whenever an oscillator's phase hits the threshold $\phi = 1$, it emits a pulse signal (fire event) and immediately resets to 0. Whenever an oscillator receives such a pulse from another oscillator, it immediately updates its own phase by performing a phase jump. The size of the phase jump is determined by the *update function* $H(\phi)$. If one oscillator i fires, the ensemble of oscillators reacts as follows (here t^+ represents an infinitesimally short time step after t):

$$\phi_i(t) = 1 \quad \Rightarrow \quad \begin{cases} \phi_i(t^+) = 0 \\ \phi_j(t^+) = H(\phi_j(t)) \quad \forall j \neq i \end{cases}, \tag{1}$$

where $H(\phi)$ is bounded by 1. It was shown by Mirollo and Strogatz [1990] that this system converges to a steady state, i.e., the phases ϕ_i of all oscillators coincide after some time. This convergence holds if $H(\phi)$ fulfills certain properties, and it holds for almost all initial conditions.

2.2. Coupling with Delays

The modeling of oscillator coupling without any delay poses an unrealistic assumption for many natural and technical systems. For example, propagation and signal processing delays do occur and cannot be ignored. If delays are present in the Mirollo-Strogatz model as described above, the system of coupled oscillators may become unstable and is in general unable to synchronize. This is because a firing oscillator may receive "echos" of its own pulse, which may cause it to fire again. This may trigger avalanches of fire events.

Kuramoto [1991] noticed these negative effects and introduced a *refractory period* t_{ref} during which an oscillator is not allowed to perform phase jumps. An oscillator enters a refractory state after each of its fire events. The application of a refractory state for each oscillator reassures an appropriate synchronization behavior. The required period length is determined by the delays occurring in the system. The use of a refractory period corresponds to the use of a refractory phase interval $[0, \phi_{\text{ref}}]$ during which phase jumps are prohibited. Assuming a fixed propagation delay τ, the updating rules can be described as follows:

$$\phi_i(t) = 1 \quad \Rightarrow \quad \begin{cases} \phi_i(t^+) = 0 \\ \phi_j(t+\tau^+) = H\big(\phi_j(t+\tau)\big) & \text{if } \phi_j(t+\tau) \notin [0, \phi_{\text{ref}}] \quad \forall j \neq i \\ \phi_j(t+\tau^+) = \phi_j(t+\tau) & \text{if } \phi_j(t+\tau) \in [0, \phi_{\text{ref}}] \quad \forall j \neq i \end{cases}. \tag{2}$$

If there are different delays between any two oscillators, with a maximum occurring delay of $\tau_{\max}$, a refractory period of $t_{\text{ref}} = 2\tau_{\max}$ ensures that no echo effects occur. This corresponds to a refractory phase $\phi_{\text{ref}} = \phi(2\tau_{\max}) = 2\tau_{\max}$.

Mathar and Mattfeldt [1996] proved that a system of two oscillators synchronizes up to some extend if we use a refractory period as defined above with excitatory coupling. The synchronizing behavior of larger sets of oscillators was indicated via simulations. Although this observation supports the applicability of coupled oscillators for technical systems, the occurrence of inhomogeneities in the phase rates $\dot{\phi}$ — as observed in real-world systems — can still lead to desynchronization (see [Tsodyks et al. 1993]). Ernst et al. [1998] and Timme et al. [2002] study the impact of delays on synchronization processes in the field of neuroscience. Timme et al. [2002] show that even a small delay disrupts the synchronized state in excitatory coupling.

2.3. Excitatory and Inhibitory Coupling

The update function $H(\phi)$ has significant impact on the synchronization behavior of the system. An update function leading to positive phase jumps (i.e. $H(\phi) \geq \phi$) was mathematically first of interest, for example in the work by Peskin [1975]. This kind of coupling is called *excitatory coupling* in the literature (see Figure 1). With the increasing interest in coupled oscillators, the impact of negative phase jumps on the dynamic system behavior got also analyzed. Such kind of coupling is called *inhibitory coupling* (see Figure 1). A publication by van Vreeswijk et al. [1994] was probably the first to realize that inhibitory coupling may support synchronization in neural systems. Ernst et al. [1995] show that inhibitory coupling has advantageous effects on the resilience to disturbances in the system.

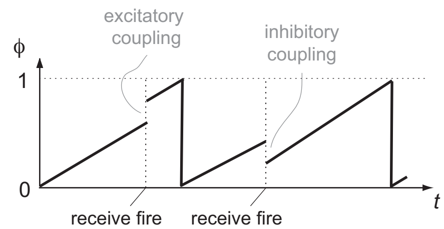


Fig. 1. Phase evolution of an oscillator without delay using excitatory or inhibitory coupling.

The main differences between excitatory and inhibitory coupling can be illustrated using the following example: Given is a set of three pulse coupled oscillators with the update function $H(\phi) = (1 + \alpha)\ \phi$ with parameter $\alpha \in (-1, 1]$. If α is positive, we have excitatory coupling. If α is negative, we have inhibitory coupling. The oscillator ϕ_0 fires. Let ϕ_1 and ϕ_2 be the two reacting oscillators, such that their phase after reaction will not hit the threshold. We are now interested in the phase difference. At pulse reception time t, their phase difference is $|\phi_1(t) - \phi_2(t)|$. After phase adjustment, at time t^+, their phase difference is

$$|\phi_1(t^+) - \phi_2(t^+)| = |(1 + \alpha)\ \phi_1(t) - (1 + \alpha)\ \phi_2(t)| = (1 + \alpha) \cdot |\phi_1(t) - \phi_2(t)|\ .$$

In other words, the phase difference between the oscillators multiplies by the factor $(1+\alpha)$. If α is positive, the oscillators' phases move further apart. If α is negative, however, the oscillators' phases move closer together. This example shows a phase combination leading to disadvantageous effects with excitatory coupling. Using inhibitory coupling, however, any phase combination decreases the phase difference between receiving oscillators, which can be seen as a positive property for synchronization.

This positive property is one of the reasons why inhibitory coupling has obtained increasing attention, especially in research on neural networks. Within this field, much effort has been taken to identify stable behaviour. Jin [2002] shows that an inhibitory-coupled system without any delays converges to some clusters. The papers [Jahnke et al. 2008; 2009] show that stable periodic fire patterns can emerge. Bloch and Romero [2002] allow oscillators to store their fire event history and show that patterns emerge which can lead to synchrony under certain conditions. Despite the increasing amount of research done in this domain, the convergence to a synchronized system state for all initial conditions could not be proven yet (see [Timme and Wolf 2008]).

The article at hand applies the general idea of inhibitory coupling — so far mainly used in neuroscience — in the area of computer and communications engineering. This application area poses different system restrictions, but it also reveals new freedom, as we can freely define the update function H. We focus on a generic group of update functions. This group also includes linear update functions, which seem to be appealing in terms of computational complexity and energy efficiency. The new application area also calls for different goals: Whereas neuroscience is often interested in stable patterns, computer and communication networks intend to achieve full system synchronization. We now extend and modify the concept of inhibitory coupling and reformulate it in such a way that it guarantees synchronization. The application now also extends to a heterogeneous system in which each oscillator can have a different phase rate $\dot{\phi}$.

3. SYNCHRONIZATION WITH INHIBITORY COUPLING

3.1. Algorithm

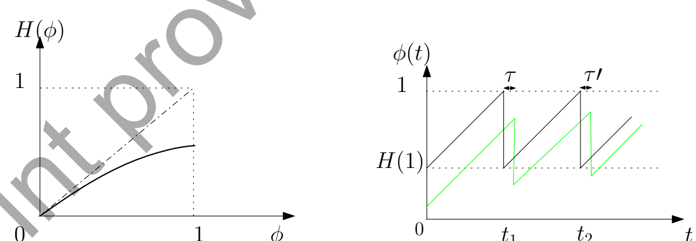
We consider a set Ω of pulse coupled oscillators. An oscillator i is represented by its phase $\phi_i(t)$ at time instant t, and its phase rate is given by $\dot{\phi}_i(t) = \kappa_i$ at all non-event times, where the constant $\kappa_i \in [1-\varepsilon, 1+\varepsilon]$ with $\varepsilon \ll 1$ denoting the maximum possible phase rate deviation. In other words, all oscillators have similar but in general non-equal rates, and an oscillator's rate remains constant over time. The phase of an oscillator i at time instant t can thus be described as

$$\phi_i(t) = \phi_i(\tilde{t}) + \kappa_i(t - \tilde{t}) \tag{3}$$

where $\tilde{t} < t$ is a previous time instant of the same cycle, unless a phase update is performed. Whenever an oscillator reaches the threshold $\phi = 1$, a fire event occurs and it immediately transmits a pulse. The time duration between this fire event at oscillator i and its complete reception at another oscillator $j \neq i$ is called delay τ_{ij}. This article considers the most general case of an inhomogeneous system with nonconstant and individual delays between any two oscillators. The maximum possible delay in the system is known and called $\tau_{\max}$.

When oscillator i fires at time t, the system behaves according to the following rules: First, each receiving oscillator $j \neq i$ adapts its phase upon reception at time $t + \tau_{ij}$ if it is not within a defined refractory interval. The phase adjustment is done via the update function $H(\cdot)$. This behavior corresponds to the standard pulse coupled oscillator with delays. Second, the sending oscillator i adapts its phase also via the update function $H(\cdot)$. Such a *self-adjustment* is not performed in the standard pulse coupled oscillator model described in (1) and (2), respectively, where a firing oscillator always resets its phase to zero. In summary, there is an instant self-adaptation of the firing oscillator and a delayed adaptation of all receiving oscillators.

In the scope of this article, the update function $H(\cdot)$ is a twice continuous differentiable function following $H(0) = 0$, $0 \leq dH(\phi)/d\phi < 1$ and $d^2H(\phi)/d\phi^2 < 0$ for all phases $\phi \in [0, 1]$. An example for such a function is given in Figure 2(a). Complying with these properties, the oscillators will always perform non-positive phase jumps, i.e., the system experiences inhibitory coupling. Note that these assumptions are slightly different than those in the Mirollo-Strogatz model.



(a) Example of an update function $H(\cdot)$.

(b) Phase evolution of two oscillators using inhibitory coupling.

Fig. 2. Update function and phase evolution with inhibitory coupling and self-adjustment.

The cycle of an isolated (non-coupled) oscillator is now $[H(1), 1]$ after the first fire event. The cycle length ω is thus $\omega = 1 - H(1)$. The behavior of two coupled oscillators can be seen in Figure 2(b). One of the oscillators reaches the threshold, fires, and performs a phase jump to $H(1)$. After some delay τ, the second oscillator receives the fire pulse and adjusts its phase as well. It may jump to a phase lower than $H(1)$. Both oscillators again increase their phases, the first oscillator fires again, and so on. In general, if we consider a system

Algorithm 1 Synchronization with inhibitory coupling and self-adjustment.

(1) An oscillator i increases its phase ϕ_i from 0 to 1 ($\forall i \in \Omega$).
(2) Whenever an oscillator i reaches $\phi_i(t) = 1$, the oscillator sends a pulse and adjusts its phase to $\phi_i(t^+) = H(1)$ (instantaneous self-coupling).
(3) The refractory phase is $\phi_{\text{ref}} = H(1) + 2(1+\varepsilon)\tau_{\max}$.
(4) Whenever an oscillator j receives a pulse from i (reception time $t+\tau_{ij}$):
  (a) It adjusts its phase to $\phi_j(t+\tau_{ij}^+) = H(\phi_j(t+\tau_{ij}))$ if $\phi_j(t+\tau_{ij}) \notin [0, \phi_{\text{ref}}]$ ($\forall j \neq i$).
  (b) It keeps its phase at $\phi_j(t+\tau_{ij}^+) = \phi_j(t+\tau_{ij})$ if $\phi_j(t+\tau_{ij}) \in [0, \phi_{\text{ref}}]$ ($\forall j \neq i$).

with several oscillators, the oscillator with the highest phase rate κ will take the lead after some time, meaning that this oscillator will always cause fire events. This oscillator is called leading oscillator or simply *leader* in the following. The cycle period of the leader is $[H(1), 1]$. All other oscillators will react to pulses from the leader and finally synchronize. The fire events of the leading oscillator are indexed in ascending order using $n \in \mathbb{N}$, and their point in time is denoted by t_n.

A refractory interval is used to cope with coupling delays. Taking into account the cycle interval $[H(1), 1]$ of the firing oscillator, a maximum delay $\tau_{\max}$, and a maximum possible phase rate $1+\varepsilon$, the refractory interval is set to $[0, \phi_{\text{ref}}]$ with

$$\phi_{\text{ref}} = \max_i \{\phi_i\,(t_n + 2\tau_{\max})\} = H(1) + 2(1+\varepsilon)\tau_{\max}\,, \tag{4}$$

with t_n being the fire instants of the leading oscillator. Oscillators whose current phase ϕ_i is within this interval when receiving a pulse do not update their phases.

The overall synchronization scheme is summarized in Algorithm 1 and denoted as *synchronization with inhibitory coupling and self-adjustment* (SISA). The main difference of SISA compared to inhibitory coupling by Timme et al. [2002] is the self-adjustment concept of firing oscillators. Although this modification appears to be minor, it tremendously changes the dynamics of the system. First, it preserves the order of the oscillators' phases if delays and rates are equal for all oscillators. This fact simplifies the mathematical analysis (see Section 3.2). Second, the positive effect of inhibitory coupling illustrated above does now apply to any fire event and any oscillator, regardless whether it just fired or adjusts. This property contributes to a straightforward mathematical analysis and is a key issue for robustness (see Section 4).

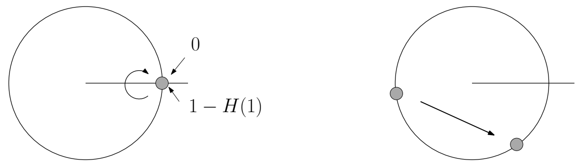
Finally, we introduce an alternative representation of an oscillator's state which takes into account the reduced cycle period. Each phase value $\phi \in [0, 1]$ is projected to a point p on a circle of length $1 - H(1)$ (see Fig. 3(a)). The mapping function is

$$p(\phi) := \big(\phi - H(1)\big) \mod \omega \quad \text{with} \quad p \in [0, \omega)\,. \tag{5}$$

The leading oscillator evolves counterclockwise on this circle in a continuous manner without any jumps. The other oscillators also evolve counterclockwise on the circle and perform jumps whenever receiving a pulse. Due to the modulo mapping of $p(\cdot)$, these infinitely short jumps can be interpreted as clockwise or counterclockwise jumps on the circle. Recall that nonleading oscillators do not send pulses.

Our goal is now to prove the synchronizing behavior of the inhibitory coupling algorithm, i.e., its convergence to a synchronized state. Furthermore, we would like to make a statement about its synchronization precision. The following section describes our approach and shows a number of properties of the algorithm that will be used later.

3.2. Prerequisites



(a) Phase change of the leading oscillator.

(b) Example of a phase jump for an adjusting oscillator.

Fig. 3. Circular phase representations with inhibitory coupling upon a fire event.

3.2.1. Definition of Precision. The collection of all phases in a set of oscillators Ω is denoted by the vector $\boldsymbol{\phi}$. As a measure for the *synchronization precision* at time t, we define

$$\Pi(t) \coloneqq \Pi(\boldsymbol{\phi}(t)) \coloneqq \max_{i,j \in \Omega} \min\left(\omega - |p(\phi_i(t)) - p(\phi_j(t))|, |p(\phi_i(t)) - p(\phi_j(t))|\right) . \quad (6)$$

If the differences $p(\phi_i(t)) - p(\phi_j(t))$ at a given time t are zero for all possible oscillator pairs, the precision is zero and thus all oscillators are running in full synchrony without any error.

This best case is in general not achieved with the given modeling assumptions, namely with delays and heterogeneous phase rates. Nevertheless, we say that the system *synchronizes up to a certain bound* Γ if we can find a value Γ, such that $\Pi(t) \leq \Gamma$ for all times $t \geq t_c$ after some convergence time t_c. Any precision bound therefore also accounts for a synchronization bound and vice versa.

3.2.2. Approach for the Convergence Proof. Given these observations, our approach to prove the synchronizing behavior is as follows. For each time instant t, an upper bound for the precision is

$$\Pi(t) \ \leq \ \max_{i,j \in \Omega} |\phi_i(t) - \phi_j(t)|. \quad (7)$$

We derive an expression for this maximum phase difference for each fire event and show that this maximum difference decreases over time (synchronizing behavior) until it stays within some precision interval (a precision limit is achieved). We derive an upper bound for this precision as a function of the system parameters ε and $\tau_{\max}$ and the update function $H(\cdot)$.

3.2.3. Notation of Time Instants and Intervals. As mentioned above, the fire events are indexed with n, and their point in time is t_n. The time that passes between a firing oscillator i sending a pulse and a receiving oscillator j reacting to the pulse is called delay τ_{ij}. The time period between two fire events is

$$\Delta t_n \coloneqq t_{n+1} - t_n \, . \quad (8)$$

The length of this period depends on the observed cycle (since the leader may change at an early stage of the observation). Whenever needed, $\tilde{t}$ represents a time variable (just as t) and $\tilde{\tau}$ a delay (just as τ). The notation t^- indicates a time instance infinitesimally short before t.

3.2.4. Three Lemmata. The following paragraphs present three lemmata describing some properties of a pulse coupled oscillator system following the inhibitory coupling algorithm. The lemmata are used later on to prove the synchronizing behavior.

LEMMA 3.1 (ADJUSTMENT PERIOD). *The time interval $[t_n, t_n + 2\tau_{\max}]$ is the time window wherein all adjustments upon a fire event of the current cycle $[t_n, t_{n+1})$ take place.*

PROOF. We have to consider three oscillator situations: the oscillator that fires and initiates a new cycle, oscillators that do not fire, and oscillators that fire but do not initiate a new cycle.

— An oscillator i fires at time t_n and initiates a new cycle. It immediately adjusts its phase to $\phi_i(t^+) = H(1)$ and enters the refractory state. It exits the refractory state at phase $\phi_i(t) = \phi_{\text{ref}}$, which happens at $t_n + 2\tau_{\max}$ the earliest. Within the refractory interval the oscillator does not adjust.
— An oscillator j does not fire and adjusts at time $(t_n + \tau_{ij})$ to a phase $\phi_j(t_n + \tau_{ij}^+) = H(\phi_j(t_n + \tau_{ij}))$. It then remains in refractory state until $\phi_j(t) = \phi_{\text{ref}}$ which happens at $t_n + 2\tau_{\max}$ the earliest.
— An oscillator k fires shortly after t_n. To do so, it has to fire before it adjusts to the fire of oscillator i at $t_n + \tau_{ik}$, when it enters the refractory interval. So if oscillator k fires at the latest possible time, it fires at $t_n + \tau_{ik}^-$, and immediately adjusts to $\phi_k(t_n + \tau_{ik}) = H(1)$ and enters the refractory interval. It leaves this interval at $\phi_i(t) = \phi_{\text{ref}}$ at time $t_n + 2\tau_{\max}$. In the worst case where $\tau_{ik} = \tau_{\max}$, oscillator k emits a signal at $t_n + \tau_{\max}^-$, which arrives at the other oscillators, again in the worst case, at $t_n + 2\tau_{\max}$. Any oscillator at that time, however, will be in the refractory interval and thus not adjust.

Summing up, due to the refractory interval any oscillator will only adjust to one fire event within a cycle $[t_n, t_{n+1})$. □

LEMMA 3.2 (SYNCHRONIZATION PRECISION BOUND). *Independent of the dynamics of the system, the synchronization precision can increase up to the value*

$$\Gamma_\tau \coloneqq (1-\varepsilon)\,\tau_{\max} + 2\,\varepsilon\,\frac{1-H(1)}{1-\varepsilon} \tag{9}$$

within the time interval of a cycle.

This lemma gives a worst case bound for the precision of a set of synchronized oscillators. Thus, we cannot guarantee a better precision of a coupled oscillator system in its steady state.

PROOF. We need to find the worst case precision that can be reached by a synchronization method under the given modeling assumptions. Consider two oscillators i and j with unknown delay τ_{ij} in between. Their phase difference $|\phi_i - \phi_j|$ may change over time due to phase jumps after fire events and due to different phase rates. Let oscillator i fire at time t_n, so that oscillator j receives the pulse signal at time $t_n + \tau_{ij}$. Oscillator j will not adjust its phase if being in refractory period at time $t_n + \tau_{ij}$. The worst case in terms of precision occurs if oscillator j fired itself at time $t_n + \tau_{ij}^-$, i.e., immediately before it received the pulse from i. In this case, the phase difference $|\phi_i - \phi_j|$ is $\kappa_i\tau_{ij}$ at time $t_n + \tau_{ij}$, and evolves to $\kappa_i\tau_{ij} + (\kappa_i - \kappa_j)(t_{n+1} - t_n - \tau_{ij})$ at time t_{n+1}. This phase difference is maximal if oscillator i has the fastest possible phase rate $\kappa_i = 1 + \varepsilon$ and oscillator j has the slowest possible phase rate $\kappa_j = 1 - \varepsilon$. This yields $(1-\varepsilon)\,\tau_{ij} + 2\varepsilon\,(t_{n+1} - t_n)$. The maximum possible delay is $\tau_{\max}$. To obtain the maximum possible interevent period $\Delta t_n = t_{n+1} - t_n$, we take the whole phase interval $[H(1), 1]$ and calculate the time it takes the slowest oscillator to run through. This yields

$$\Delta t_n \le \frac{1-H(1)}{1-\varepsilon}\,. \tag{10}$$

The derivation so far considered two arbitrary oscillators and thus also holds for the ones with maximum difference. Combining these considerations, we formulate

$$\max_{i,j\in\Omega} |\phi_i - \phi_j| \;\le\; (1-\varepsilon)\,\tau_{\max} + 2\,\varepsilon\,\frac{1-H(1)}{1-\varepsilon}\,. \tag{11}$$

This bound can thus actually be reached, and Γ_τ can be defined as in (9). □

LEMMA 3.3 (LEADING OSCILLATOR). *The leader in a system of pulse coupled oscillators, with highest phase rate, remains leader.*

PROOF. Consider the oscillator with the highest phase rate in a set of oscillators. This oscillator is called i in the following; it has a phase rate κ_i and phase $\phi_i(t)$. Upon reaching the threshold 1 at time t_n, it fires and self-adjusts to $\phi_i(t_n^+) = H(1)$. At time $t_n + \tau_{\max}$ all oscillators will have adjusted. There are two different adjustment reasons. First, oscillators may adjust due to their reaction to the fire pulse from oscillator i. Such an oscillator j is not in the refractory phase at reception time $t_n + \tau_{ij}$, i.e., its phase ϕ_j before adjusting follows $\phi_{\text{ref}} < \phi_j(t_n + \tau_{ij}) < 1$. The phase after adjustment follows $H(\phi_{\text{ref}}) < \phi_j(t_n + \tau_{ij}^+) < H(1) < 1$ with inhibitory coupling. The phase of the firing oscillator i at this time is $\phi_i(t_n + \tau_{ij}^+) = H(1) + \tau_{ij}\kappa_i > H(1)$. Thus, $\phi_i > \phi_j$ is ensured, and oscillator i remains in the lead as it has the highest phase rate. Second, alternatively, the oscillator j fired itself before receiving the fire pulse from i and performed a self-adjustment to $\phi_j(\tilde{t}) = H(1)$, where $\tilde{t}$ denotes the time instance of the adjustment with $\tilde{t} \in (t_n, t_n + \tau_{ij}]$. The phase of oscillator i at this time is $\phi_i(\tilde{t}) = H(1) + \kappa_i \left(\tilde{t} - t_n\right)$. Again, $\phi_i > \phi_j$ is ensured and oscillator i stays in the lead. $\square$

3.3. Synchronization Convergence for Two Oscillators

We can now show that the inhibitory coupling algorithm performs a precision contraction from any initial conditions, and prove that oscillators synchronize up to a certain bound. We first restrict our setup to two coupled oscillators. Later we extend the analysis to an arbitrarily large set of oscillators.

THEOREM 3.4 (UPPER BOUND OF PRECISION FOR TWO OSCILLATORS). *Two oscillators with inhibitory coupling & self-adjustment synchronize up to a precision* $\Pi(t) \leq \Gamma_2$ *with*

$$\Gamma_2 \coloneqq (1+\varepsilon)\,\tau_{\max} + \frac{2\varepsilon}{1 - H'_{\max}} \cdot \frac{1 + H(1)}{1 - \varepsilon}\,. \tag{12}$$

PROOF. We consider two oscillators i and j with phase rates κ_i and κ_j and arbitrary initial phases. To show an improvement of the synchronization precision over time, we consider (7) and prove that the oscillators' phase difference between two consecutive fire events t_n and t_{n+1} decreases, i.e.,

$$|\phi_i(t_{n+1}) - \phi_j(t_{n+1})| \;<\; |\phi_i(t_n) - \phi_j(t_n)|\,, \tag{13}$$

until a certain phase difference at a certain n is achieved and thus a steady state is reached.

The evolution of a phase is linear over time for all non-event times and is given by (3). Hence, the phase difference at time t_{n+1} can be expressed by the phases at a previous time instant $\tilde{t} < t_{n+1}$ as follows:

$$|\phi_i(t_{n+1}) - \phi_j(t_{n+1})| = \left|\phi_i(\tilde{t}) - \phi_j(\tilde{t}) + (\kappa_i - \kappa_j)\left(t_{n+1} - \tilde{t}\right)\right| \tag{14}$$

for $\tilde{t} \in (t_n + \tau_{ij}, t_{n+1})$ as explained in Lemma 3.1.

Let us consider the evolution of the two phases over time (see Table I). Without loss of generality, we assume that a fire event occurs at time t_n in oscillator i, so that $\phi_i(t_n) = 1$ and $\phi_i(t_n^+) = H(1)$. The phase at this time instant at oscillator j can be written as $\phi_j(t_n) = 1 - c$ with $c \in [0, 1]$. Oscillator j receives the fire pulse after a transmission delay τ_{ij}. During that time period, its phase evolved at phase rate κ_j to $\phi_j(t_n + \tau_{ij}) = \phi_j(t_n) + \kappa_j\tau_{ij}$. Upon reception of the fire pulse, it adjusts its phase to $\phi_j(t_n + \tau_{ij}^+) = H(1 - c + \kappa_j\tau_{ij})$. The phase of the firing oscillator i evolved with rate κ_i during this time period and yields $\phi_i(t_n + \tau_{ij}^+) = H(1) + \kappa_i\tau_{ij}$. Given this, substituting $\tilde{t}$ by $t_n + \tau_{ij}^+$ in (14), the right hand side (rhs) becomes

$$|H(1) + \kappa_i \tau_{ij} - H(1 - c + \kappa_j \tau_{ij}) + (\kappa_i - \kappa_j)(t_{n+1} - t_n - \tau_{ij})| \tag{15}$$

$$= |H(1) - H(1 - c + \kappa_j \tau_{ij}) + (\kappa_i - \kappa_j)\Delta t_n + \kappa_j \tau_{ij}| \tag{16}$$

with $\Delta t_n = t_{n+1} - t_n > 0$.

Table I. Phase evolution of oscillators i and j

	ϕ_i	ϕ_j
t_n:	1	$1-c$
t_n^+:	$H(1)$	$1-c$
$t_n + \tau_{ij}$:	$H(1) + \kappa_i \tau_{ij}$	$1 - c + \kappa_j \tau_{ij}$
$t_n + \tau_{ij}^+$:	$H(1) + \kappa_i \tau_{ij}$	$H(1 - c + \kappa_j \tau_{ij})$

Let us now make use of the mean value theorem. It states that there is a phase ξ in the interval $[1 - c + \kappa_j \tau_{ij}, 1]$ with

$$H'(\xi) = \frac{dH(\xi)}{d\xi} = \frac{H(1) - H(1 - c + \kappa_j \tau_{ij})}{c - \kappa_j \tau_{ij}} \,. \tag{17}$$

Using this expression, (16) becomes

$$|H'(\xi)(c - \kappa_j \tau_{ij}) + (\kappa_i - \kappa_j)\Delta t_n + \kappa_j \tau_{ij}| \,. \tag{18}$$

We demand that (13) holds, observe that in this case the rhs of (13) equals c, and obtain the condition

$$|H'(\xi)c - H'(\xi)\kappa_j \tau_{ij} + (\kappa_i - \kappa_j)\Delta t_n + \kappa_j \tau_{ij}| \; < \; c \;. \tag{19}$$

To dissolve the absolute value, we have to consider two cases. First, assume that the expression within the absolute value bars on the left hand side (lhs) of (19) is positive. This means that the oscillators do not change order, i.e., no overtaking is performed but oscillator i stays leader. Solving (19) without the absolute value bars for c yields

$$\kappa_j \tau_{ij} + \frac{(\kappa_i - \kappa_j)\Delta t_n}{1 - H'(\xi)} \; < \; c \,. \tag{20}$$

Singularities cannot occur, since $H'(\phi) < 1$ holds independently of ϕ. As long as this inequality is fulfilled, the phase difference between two consecutive fire events decreases. This phase contraction ceases once both sides of (20) are equal. Thus, the lhs of (20) is a lower bound for the phase difference needed to achieve a phase contraction in the subsequent cycle.

Second, we assume that the expression within the absolute value bars on the lhs of (19) is negative. This means that the oscillators change order, i.e., oscillator j overtakes i. We specifically conclude $\kappa_j > \kappa_i$. This yields

$$\frac{-\kappa_j \tau_{ij}(1 - H'(\xi)) + |\kappa_i - \kappa_j|\Delta t_n}{1 + H'(\xi)} \; < \; c \,. \tag{21}$$

Singularities cannot occur, since $H'(\phi) \geq 0$ holds independently of ϕ. The same statements concerning phase contractions can be made as in the first case. Overall, a contraction can be guaranteed as long as the phases fulfill (20) and (21).

Now we generalize the derived inequalities (20) and (21) to hold for any possible case, including the worst case. We will then be able to guarantee the contracting dynamics of the system for arbitrary initial conditions as long as the inequalities hold. The parameter combinations turning the inequalities into equalities serve as synchronization bounds. For generalizing, we take into account the maximum delay $\tau_{\max} \geq \tau_{ij}$ and the maximum possible phase rate difference $|\kappa_i - \kappa_j| \leq 2\varepsilon$. We apply the upper bound (10) for Δt_n and define the maximum slope $H'_{\max} := \max_\xi H'(\xi)$ and the minimum slope $H'_{\min} := \min_\xi H'(\xi)$ of the update function. An upper bound for the lhs of (20) is given by Γ_2 as shown in (12), and an upper bound for the lhs of (21) is

$$\tilde{\Gamma}_2 := \frac{2\varepsilon}{1 + H'_{\min}} \cdot \frac{1 + H(1)}{1 - \varepsilon} \,. \tag{22}$$

A phase contraction in the form of (13) is given at least as long as Γ_2 and $\tilde{\Gamma}_2$ are smaller than the phase difference $|\phi_i(t_n) - \phi_j(t_n)|$ of the previous cycle.

Finally, we note that $\Gamma_2 > \tilde{\Gamma}_2$, since $1 - H'_{\max} < 1 + H'_{\min}$. Hence, the term Γ_2 is an upper bound for the phase difference of the two oscillators at which no further contraction can be guaranteed. Until the phase difference of Γ_2, Lemma 3.3 guarantees that we can apply this analysis repeatedly for t_m, $m > n$, $m \in \mathbb{N}$. In overall, using (7), we can finally state that

$$\Pi(t) \;\leq\; \Gamma_2 \,, \tag{23}$$

thus proving the theorem. It is not possible for any phase difference to jump above Γ that is already below. This is due to the monotonic increasing and continuous function $H(\cdot)$, and the fact that we already consider the maximal possible drift due to different phase rates. $\square$

We note that $\Gamma_2 \geq \Gamma_\tau$, which is due to Lemma 3.2.

3.4. Synchronization Convergence for Arbitrarily Many Oscillators

Let us extend our synchronization theorem and the corresponding bound to a system with more than two oscillators. Almost the same statement can be made.

THEOREM 3.5 (UPPER BOUND OF PRECISION). *A system of oscillators with inhibitory coupling and self-adjustment synchronizes up to a precision*

$$\Pi(t) \;\leq\; \Gamma \,, \tag{24}$$

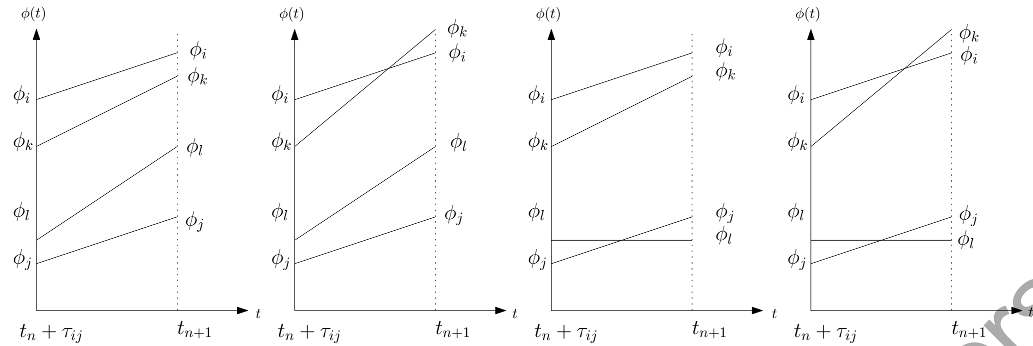
where the bound Γ *is given by*

$$\Gamma := \frac{\left(1 + \varepsilon - H'_{\min}\,(1 - \varepsilon)\right)\tau_{\max} + 2\varepsilon\,\frac{1 + H(1)}{1 - \varepsilon}}{1 - H'_{\max}} \,. \tag{25}$$

This theorem is a positive answer to our first question posed in the introduction of this article, namely whether inhibitory coupling can lead to synchronization of all oscillators independent of the initial conditions.

As a special case, the theorem also proves that delay-free homogeneous oscillators ($\tau_{\max} = 0, \varepsilon = 0$) synchronize with precision $\Pi = 0$ from any initial condition.

PROOF. We generalize the proof of Theorem 3.4 for a system of more than two oscillators, i.e., we apply the used argumentation for the whole set Ω. To account for the precision of the system we use (7) and point out that the maximum phase difference is determined by two oscillators, namely the leading oscillator and the hindmost oscillator at a given time instant t. Again, we study the change of the phase difference within one cycle, i.e., from t_n to t_{n+1}, but now we must consider the maximum phase difference over all oscillators. It is important to note that both the leading and hindmost oscillator may in general change within a cycle due to overtaking events with other oscillators. The leading oscillator may be overtaken by a faster oscillator, and the hindmost oscillator may overtake a slower oscillator. All possible cases can be described with a set of four oscillators as illustrated in Figures 4 and 5. In all cases, the leading oscillator at time t_n is called i, and the hindmost oscillator at time t_n is called j. The other oscillators are called k and l.



(a) no change (b) change of leading oscillator (c) change of hindmost oscillator (d) change of hindmost and leading oscillator

Fig. 4. Phase evolutions where the leading oscillator is not overtaken by the hindmost oscillator.

Figure 4 shows phase evolutions in which oscillator i is not overtaken by oscillator j. Figure 4(a) shows the simplest case, where — although the oscillators may have different phase rates — oscillator i remains leader, and oscillator j remains hindmost oscillator at time t_{n+1}. Hence, the evolution of the maximum phase difference is determined by these two oscillators and we can apply Theorem 3.4 with the synchronization bound Γ_2.

In Figure 4(b), the leader changes, as oscillator i is overtaken by a faster oscillator k ($\kappa_k > \kappa_i$). Thus, to compare the maximum phase differences at time t_n and t_{n+1}, we have to compare $|\phi_i(t_n) - \phi_j(t_n)|$ with $|\phi_k(t_{n+1}) - \phi_j(t_{n+1})|$. If more than one oscillators overtakes i, we consider the one leading at t_{n+1}. Denoting the time instant of the overtaking event by $\tilde{t}$, then $\phi_k(\tilde{t}) - \phi_i(\tilde{t}) = 0$ holds, and we have

$$\left|\phi_k(\tilde{t}) - \phi_j(\tilde{t})\right| = \left|\phi_i(\tilde{t}) - \phi_j(\tilde{t})\right| \ . \tag{26}$$

This phase difference changes over time due to different phase rates, and yields at time t_{n+1} the expression

$$|\phi_k(t_{n+1}) - \phi_j(t_{n+1})| = \left|\phi_i(\tilde{t}) - \phi_j(\tilde{t})\right| + (\kappa_k - \kappa_j) \cdot \left(t_{n+1} - \tilde{t}\right) \ . \tag{27}$$

If $\tilde{t} > t_n + \tau_{ij}$ we can substitute $\tilde{t}$ for t_{n+1} in (18) to state

$$|\phi_i(\tilde{t}) - \phi_j(\tilde{t})| = H'(\xi)\,(c - \kappa_j \tau_{ij}) + \kappa_i \tau_{ij} + (\kappa_i - \kappa_j) \cdot (\tilde{t} - t_n - \tau_{ij}) \,. \tag{28}$$

If we now exchange κ_i by κ_k, where $\kappa_k > \kappa_i$, we obtain an upper bound for (28). Combining (27) and (28) yields

$$|\phi_k(t_{n+1}) - \phi_j(t_{n+1})| \le H'(\xi)\,(c - \kappa_j \tau_{ij}) + \kappa_i \tau_{ij} + (\kappa_k - \kappa_j) \cdot (t_{n+1} - t_n - \tau_{ij}) \ . \tag{29}$$

If $\tilde{t} \le t_n + \tau_{ij}$, using the phase difference at $t_n + \tau_{ij}$ shown in Table I and the argument from (27), we derive

$$|\phi_i(\tilde{t}) - \phi_j(\tilde{t})| = H'(\xi)\,(c - \kappa_j \tau_{ij}) + \kappa_i \tau_{ij} - (\kappa_k - \kappa_j) \cdot (t_n + \tau_{ij} - \tilde{t}) \,. \tag{30}$$

The term $(\kappa_k - \kappa_j) \cdot (t_n + \tau_{ij} - \tilde{t})$ is always positive, thus we can bound (30) by (29) and get the contraction condition

$$\frac{\left(\kappa_i - H'(\xi)\,\kappa_j\right)\tau_{ij} + (\kappa_k - \kappa_j)\,(\Delta t_n - \tau_{ij})}{1 - H'(\xi)} \ < \ c \,. \tag{31}$$

We again make worst case assumptions to give a synchronization bound in this case. Using the same argumentation as in the proof of Theorem 3.4, we obtain Γ as given in (25) as an

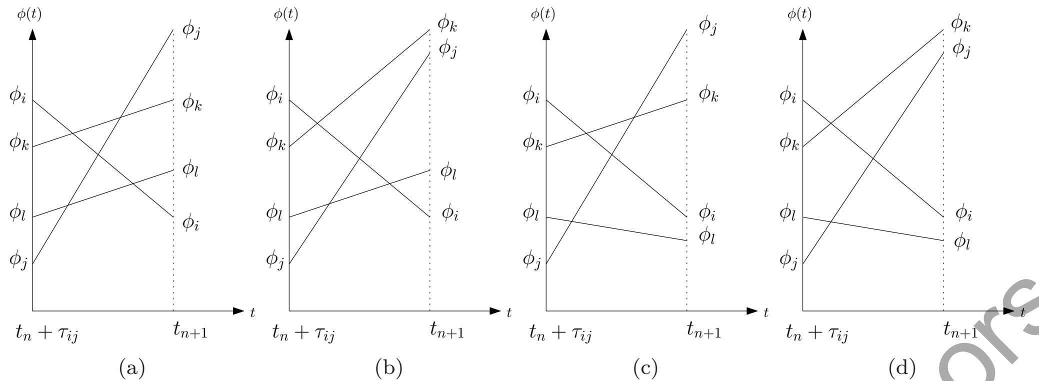


Fig. 5. Phase evolutions where the leading oscillator is overtaken by the hindmost oscillator.

upper bound for the lhs of (31). The same argumentation holds for a change of the hindmost oscillator (see Figure 4(c)), and thus for a combination of both (see Figure 4(d)). Thus, also in these cases, the bound Γ holds.

Figure 5 shows phase evolutions in which oscillator i is overtaken by oscillator j. In Figure 5(a), the additional oscillators do not influence the phase difference at time t_{n+1}, which enables us to apply the results for two oscillators with Γ_2. Figure 5(b) shows a change in the leading oscillator. The supposed leader j is overtaken by oscillator k, who becomes the leader at t_{n+1}. This case is similar to that of Figure 4(d). In both cases the leading and hindmost oscillators i and j are exchanged by others namely, (k and l) and (k and i), respectively. Thus the argumentation of Figure 4(d) can be applied by exchanging the oscillators' names. Figures 5(b) and 5(d) follow the same argumentation. By the exchange of the leading and hindmost oscillator we can use the considerations of Figure 4(d). We rename the oscillators at t_{n+1} and apply the same argument as before to obtain the bound Γ.

Comparing (12) and (25) we observe that $\Gamma > \Gamma_2$. Hence, for the general case with an arbitrary number of oscillators, the bound Γ as given in (25) holds for all possible cases, leading to Theorem 3.5. $\square$

4. PERFORMANCE AND ROBUSTNESS

This section gives a simulation-based performance study of inhibitory coupling with self-adjustment (SISA) in terms of synchronization precision and makes a comparison with the performance of standard excitatory coupling. As a byproduct, we observe how close the analytical bound Γ comes to the simulated values of the synchronization precision. Furthermore, recall from the introduction of this article that one of the motivations to consider inhibitory coupling for synchronization is to improve the robustness against faulty behavior. To study this issue, we analyze the impact of false fire events and failure of fire detection on the synchronization precision, showing that the proposed algorithm can have positive effects on the precision for appropriate parameter choices. Such faults in networked systems may occur, for example, due to defective oscillators or malicious members that intrude into the system or by errors on the communication channel. In more detail, three fault scenarios are modeled: a single erroneous fire event inserted at a random point in time; a series of erroneous fire events randomly distributed over a specific time interval; and probabilistic failures of fire detection. The reaction to such disturbances is shown in order to demonstrate the capability of the system to recover. The results give an answer to our second question posed in the introduction as to whether inhibitory coupling can improve the robustness against certain faults. Note that various types of errors are investigated in the literature of self-organizing synchronization. For example, Hong and Scaglione [2005] take into account fire detection in noisy environments; Babaoglu et al. [2007] study the impact of churn and message loss; and Tyrrell et al. [2010] analyze missed and false fire detection.

4.1. Definition of Normalized Precision

A fair comparison of excitatory and inhibitory coupling via the representative algorithms given in Sections 2.2 and 3.1 requires a normalization in terms of time periods and an alignment in the definition of precision. As excitatory coupling uses the cycle interval $[0, 1]$ while inhibitory coupling uses $[H(1), 1]$, also the duration of a cycle is in general different. Comparing absolute time periods (e.g. the time-to-synchrony) reveals an inherent advantage to inhibitory coupling as its cycle period is shorter. A fairer approach is to compare the number of cycles rather than the absolute time (i.e. measure the cycles-to-synchrony).

We also need to consider the difference in cycle length when defining a synchronization precision. Denoting the length of the cycle interval by ω, a normalized precision is

$$\Pi^\star(t) := \frac{1}{\omega}\,\Pi(t)\,, \tag{32}$$

such that $0 \leq \Pi^\star(t) \leq 0.5$. The worst precision 0.5 can be interpreted as the maximum distance between two points on a circle of circumference 1. Using inhibitory coupling, the cycle length is $\omega = 1 - H(1)$ and the nonnormalized precision $\Pi(t)$ is defined in (6). Consequently, the normalized precision bound for inhibitory coupling with self-adjustment is $\Gamma^\star := \frac{\Gamma}{1-H(1)}$ with Γ given by (25). Using excitatory coupling, the cycle length is $\omega = 1$ and Expression (6) for the precision simplifies to

$$\Pi(t) := \max_{i,j\in\Omega} \min(1 - |\phi_i - \phi_j|, |\phi_i - \phi_j|)\,, \tag{33}$$

which is obtained using (5) with the reset value $H(1) = 0$ upon a fire event.

The precision of both systems is measured at the end of each cycle n infinitesimally before a fire event takes place and a new cycle starts (i.e., at time t_n). We also make sure that no fire event is still in translation at this time. The normalized precision of the system in its steady state is called $\overline{\Pi}^\star$; it is calculated as the mean of the precision values sampled at 15 subsequent cycles in steady state. A system is considered to be in steady state if the precision remains constant over time despite a negligible precision variation. If the normalized precision in steady state is below the threshold $\Gamma^\star$, the system is also in synchronized state.

4.2. Simulation Setup

Table II shows the used simulation parameters. We employ the linear update function $H(\phi) = (1 + \alpha)\,\phi$ with coupling strength $\alpha \in (-1, 1)$, which fulfills the requirements from Section 3.1. If α is set positive, we have excitatory coupling. If α is set negative, we have inhibitory coupling. The cycle length yields $\omega = 1$ for excitatory coupling and $\omega = |\alpha|$ for inhibitory coupling. The following simulations study two different coupling strengths: a strongly-coupled system with $|\alpha| = 0.99$ and a weaker coupled system with $|\alpha| = 0.5$. The same weaker coupled system was used by Rhouma and Frigui [2001].

The phase rate κ of an oscillator is chosen randomly following a uniform distribution on the interval $[1 - \varepsilon, 1 + \varepsilon]$. The maximum phase rate deviation is $\varepsilon = 0.005$. We simulate a system of 10 oscillators, where each oscillator starts with a random initial phase.

The delay between two oscillators is chosen uniformly at random between a minimum delay $\tau_{\min}$ and a maximum delay $\tau_{\max}$. The minimum delay is 1 % and the maximum delay is 4 % of the cycle duration assuming excitatory coupling with $\varepsilon = 0$. The delay values for inhibitory coupling are scaled by a factor of ω due to different cycle lengths. To account for these delays, each oscillator employs a refractory state. The refractory phase should be

Table II. Simulation parameters

Parameter	Value
Update function $H(\phi)$	$(1+\alpha)\,\phi$
Coupling strength $\alpha \in (-1, 1)$	$\lvert\alpha\rvert \in \{0.50, 0.99\}$
Maximum phase rate deviation ε	0.005
Number of oscillators	10
Delay $[\tau_{\mathrm{min}}, \tau_{\mathrm{max}}]$	$[1\,\%, 4\,\%]$
Refractory phase ϕ_{ref}	
- Excitatory	0.081
- Inhibitory with $\alpha = -0.99$	0.091
- Inhibitory with $\alpha = -0.50$	0.550

larger than $\phi_{\mathrm{ref}} = 2\,(1+\varepsilon)\,\tau_{\mathrm{max}} = 0.0804$ with excitatory coupling; it should be larger than $\phi_{\mathrm{ref}} = 1 - \lvert\alpha\rvert + 2\,(1+\varepsilon)\,\omega\,\tau_{\mathrm{max}} = 1 - 0.9196\,\lvert\alpha\rvert$ with inhibitory coupling. Given these expression, the values listed in Table II are used.

To make a statement about the expected system performance, 1000 simulations are made, from which the mean value and standard deviation of the precision values $\Pi^\star(t)$ are computed for each cycle.

4.3. Synchronization Performance

Let us first study the synchronizing behavior of both coupling techniques from a random initial condition to a steady state after some time and also measure the synchronization precision achieved in this state.

Figure 6 shows the evolution of the expected normalized synchronization precision $\Pi^\star$ and its sample standard deviation over time. Time is measured in terms of the number of cycles. Both approaches converge to some extent to a steady state with a synchronization precision $\Pi^\star$ below 0.05 and thus achieve a synchronized state. Full synchrony ($\Pi^\star = 0$) is impossible due to the varying delays. A synchronized state is reached within only a few cycles depending on the coupling strength α. Clearly, a higher coupling strength $\lvert\alpha\rvert$ leads to faster convergence. For the given scenarios, the time-to-synchrony is about four cycles for excitatory coupling, and between four and nine cycles for inhibitory coupling. This means that inhibitory coupling does not provide a faster convergence than excitatory coupling.

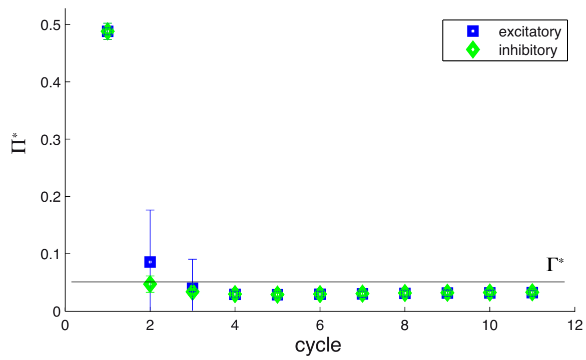
We also observe that the synchronization precision is better than the upper bound $\Gamma^\star$ (also see Table III). The bound is rather tight for high $\lvert\alpha\rvert$ but very loose for low $\lvert\alpha\rvert$. Table III summarizes the mean precisions $\overline{\Pi}^\star$ achieved in synchronized state. It also shows that the bound $\Gamma^\star$ is actually very close to the minimal possible bound $\Gamma^\star_\tau$.

Table III. Mean precision in synchronized state and precision bounds

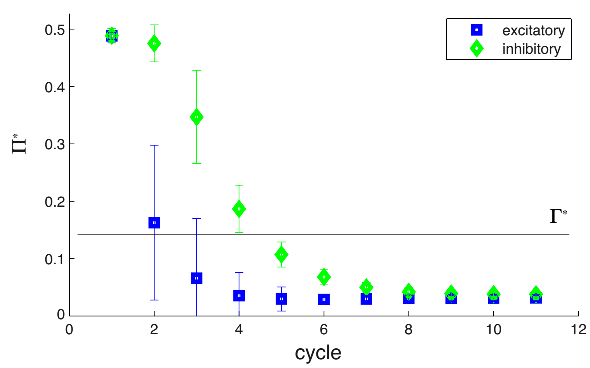
	Excitatory	Inhibitory coupling		
Coupling strength	$\overline{\Pi}^\star$	$\overline{\Pi}^\star$	$\Gamma^\star$	$\Gamma^\star - \Gamma^\star_\tau$
(a) $\lvert\alpha\rvert = 0.99$	0.034	0.035	0.051	$7 \cdot 10^{-4}$
(b) $\lvert\alpha\rvert = 0.5$	0.034	0.041	0.142	$5 \cdot 10^{-2}$

4.4. Robustness

We now investigate the reaction of a coupled oscillator system in its synchronized state when being disturbed. Our goal is to study how robust the system is in terms of its synchroniza-



(a) Coupling strength $\alpha_{\text{excitatory}} = 0.99$ and $\alpha_{\text{inhibitory}} = -0.99$



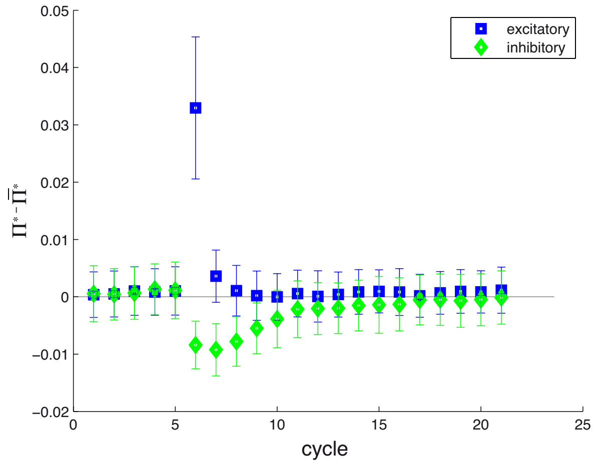
(b) Coupling strength $\alpha_{\text{excitatory}} = 0.5$ and $\alpha_{\text{inhibitory}} = -0.5$

Fig. 6. Evolution of synchronization precision $\Pi^{\star}(t)$ from random initial conditions.

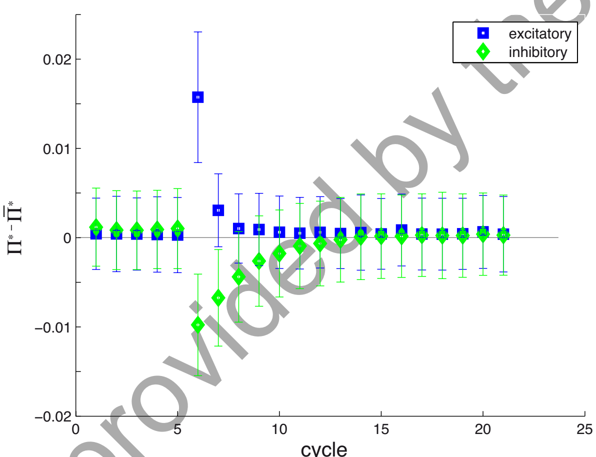
tion precision, and whether there are differences between the robustness of excitatory and inhibitory coupling.

4.4.1. Single Random Fire. We first investigate the precision after a signal false fire event at a random point in time. We start the simulation with a random initial condition, let it run until the system reaches a synchronized state, then broadcast a fire pulse, and finally measure its influence on the precision in the following cycles. The fire event happens randomly within the phase interval $[0, 1)$ for excitatory coupling and within $[H(1), 1)$ for inhibitory coupling.

Figure 7 shows the evolution of the precision disturbance $\Pi^{\star}(t) - \overline{\Pi}^{\star}(t)$ over time, where the system is already in synchronized state at cycle 0 and the fire event occurs in cycle 6. If the coupling is strong with $|\alpha| = 0.99$ as shown in Figure 7(a), the disturbance of excitatory coupling is most prevalent. Its precision deviates by about 0.03 on average but then quickly



(a) Coupling strength $\alpha_{\text{excitatory}} = 0.99$ and $\alpha_{\text{inhibitory}} = -0.99$



(b) Coupling strength $\alpha_{\text{excitatory}} = 0.5$ and $\alpha_{\text{inhibitory}} = -0.5$

Fig. 7. Precision disturbance from synchronized state as a reaction to a false fire at cycle 6.

recovers to its synchronized state. Inhibitory coupled systems are expected to deviate in a less severe manner; the average disturbance is here about 0.01. In fact, the random fire pulse temporarily improves the precision for a short time period; the steady state is then regained within a few cycles.[1] Figure 7(b) shows the performance of a weaker coupled system with $|\alpha| = 0.5$. The behavior after a fire pulse is similar in this case. Overall, excitatory coupling recovers faster, but inhibitory coupling deviates less from the mean precision and its deviation is actually improving the precision. The choice of the coupling parameter α

[1]We interpret the jump in precision for the inhibitory coupling by the gained adjustment. An additional adjustment improves the precision, therefore we see an instantaneous improvement. Since the adjustment frequency in the following stabilizes again we observe a convergence to the steady state thereafter.

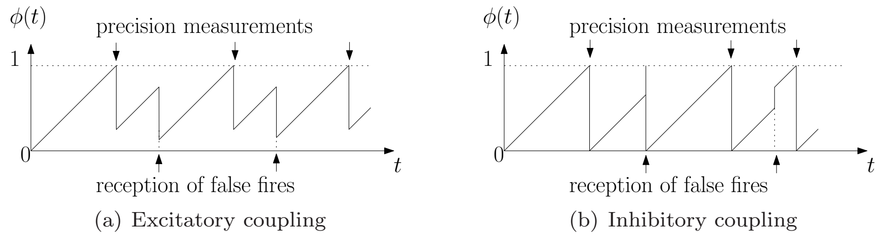


Fig. 8. Precision measurement points

does not significantly change the behavior (note that $\alpha = -0.99$ is close to the boundary of possible values).

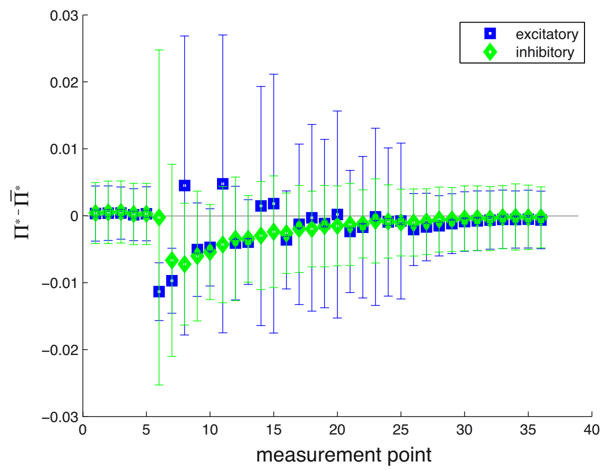
4.4.2. Repeated Random Fire. We now investigate the impact of repeated fire events on the precision of coupled oscillators. Figure 8 illustrates the measurement points for the synchronization precision. Measurements are made at the end of each cycle unless its end is caused as an immediate reaction to a false fire (see Figure 8(a)). A single false fire event is injected between two measurement points. This approach ensures a fair comparison between inhibitory and excitatory coupling. It would be unfair to allow the inhibitory-coupled system to have several adjustments between two measurement points. The time instants of false fire events are sampled from uniform random distribution between two measurement points. A fire pulse is neglected by a receiving oscillator if the oscillator is in refractory state. Hence, it is possible that for some cycles practically no false fire pulse is experienced between two measurement points. On the contrary, due to the restriction of no fires in translation, it is also possible to receive several false fire pulses between two measurement points.

Given a system of coupled oscillators in synchronized state, we inject a series of random fire events for a duration of 20 measurement periods. We observe the system precision during and after this period. Figure 9 shows the resulting disturbance in precision over time. A high coupling strength $|\alpha| = 0.99$ yields the following result (see Figure 9(a)): Inhibitory coupling shows a smooth evolution of the mean precision. Excitatory coupling leads to a fluctuating mean precision. Furthermore, the standard deviation of the precision for excitatory coupling exceeds that of inhibitory coupling. The overall plot gives rise to the interpretation that inhibitory coupling is somewhat more robust in this scenario. As for the single false fire, the precision of inhibitory coupling temporarily improves and then smoothly stabilizes again.[2] Overall, both coupling schemes can cope with the additional fire pulses and regain the synchronized state.

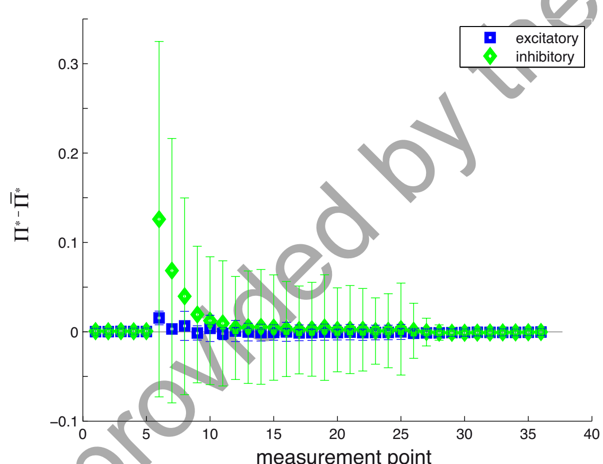
A system with weak coupling, as given in Figure 9(b) with $|\alpha| = 0.5$, shows a completely different behavior. Here, the disturbance of the excitatory-coupled system is very low, especially compared to inhibitory coupling. Both mean and standard deviation of the precision degrade only slightly. With inhibitory coupling, though, the mean precision disturbance increases up to 0.12 and slowly recovers. In both coupling schemes, the steady state for the mean precision is regained even while random fires are present. Nevertheless, the standard deviation of inhibitory coupling is much higher. Once the random fire period finishes, both algorithms reach their synchronized state again.

4.4.3. Failure of Fire Detection. Let us finally assume that oscillators are sometimes unable to detect fire events. Such failures occur in communication systems due to temporarily bad channel conditions caused by fading or interference (also see [Tyrrell et al. 2010]). To be more specific, we simulate each received pulse to be lost with a certain probability $q = 2\,\%$, $5\,\%$, or

[2] The temporal improvement can be explained by the increased update frequency. Due to multiple updates, the deviation of phases due to different phase rates cannot evolve as strong as before.



(a) Coupling strength $\alpha_{\text{excitatory}} = 0.99$ and $\alpha_{\text{inhibitory}} = -0.99$



(b) Coupling strength $\alpha_{\text{excitatory}} = 0.5$ and $\alpha_{\text{inhibitory}} = -0.5$

Fig. 9. Precision deviation from synchronized state as a reaction to false fires at cycles $\{6, 7, \ldots, 25\}$.

10 %. The results are as follows. Qualitatively speaking, the mean synchronization precision performs a smooth transition from its starting value to a steady state below the bound $\Gamma^\star$. The steady state precisions are in the same order of magnitude as those with perfect channel conditions, but the time-to-synchrony increases compared to perfect channel conditions. A synchronized state is achieved in about five cycles ($q = 2\,\%$), seven cycles ($5\,\%$), and nine cycles ($10\,\%$), respectively, for $\alpha = -0.99$. This behavior demonstrates a certain robustness against failures in detecting a fire.

5. CONCLUSIONS

This article introduced a self-organizing synchronization scheme with inhibitory coupling and self-adjustment. Such pulse coupled oscillator systems are adaptive and scalable, and can be applied to dynamic networked systems.

It was proven (for the first time) that inhibitory coupling can lead to global synchronization independent of initial conditions for delay-free and homogeneous oscillators. Furthermore, we extended the theory of delayed coupled oscillators with inhomogeneous delays and inhomogeneous phase rates. It was proven that such systems synchronize up to a certain precision bound Γ, and we derived this bound for arbitrarily large systems of oscillators. We can thus guarantee that the introduced SISA algorithm synchronizes for arbitrary many inhomogeneous oscillators with varying transmission delays. Simulation results show that the bound Γ gives a good estimate in strongly coupled systems. Comparing inhibitory coupling with excitatory coupling showed that inhibitory coupling can yield the same precision.

It was further shown that systems of coupled oscillators display a certain level of robustness against randomly injected false fire events and missed fire events, illustrating a certain level of resilience against faulty or malicious members in natural and technical systems and against failures of the communication medium. In all cases investigated in this article, the system was able to regain a synchronized state within a few cycles, sometimes even during the disruption period. The specific level of disruption in the synchronization precision, however, strongly depends on the system parameters. The scenarios of this article indicate that, for specific conditions, inhibitory coupling can be more robust than excitatory coupling especially for strong coupling.

ACKNOWLEDGMENTS

The authors would like to thank Wilfried Elmenreich (University of Klagenfurt) and Marc Timme (Max Planck Institute for Dynamics and Self-Organization) for fruitful discussions and interesting views on the topic of this paper. The authors would also like to thank the three anonymous reviewers for several useful comments. A special thanks goes to Kornelia Lienbacher and Günther Brandner (both University of Klagenfurt) for proofreading.

REFERENCES

ABBOTT, L. F. AND VAN VREESWIJK, C. 1993. Asynchronous states in neural networks of pulse-coupled oscillators. *Phys. Rev. E 48,* 2, 1483–1490.

BABAOGLU, O., BINCI, T., JELASITY, M., AND MONTRESOR, A. 2007. Firefly-inspired heartbeat synchronization in overlay networks. In *Proc. Intern. Conf. Self-Adaptive Self-Organ. Sys.* Boston, MA, 77–86.

BLOCH, I. J. M. AND ROMERO, C. 2002. Firing sequence storage using inhibitory synapses in networks of pulsatil nonhomogeneous integrate-and-fire neural oscillators. *Phys. Rev. E 66,* 3, 036127.

BUCK, J., BUCK, E., CASE, J., AND HANSON, F. 1981. Control of flashing in fireflies. V. Pacemaker synchronization in *pteroptyx cribellata. J. Comp. Physiology A 144,* 3, 630–633.

CAMAZINE, S., J.-L.DENEUBOURG, FRANKS, N. R., SNEYD, J., THERAULAZ, G., AND BONABEAU, E. 2001. *Self-Organization in Biological Systems.* Princeton University Press.

ERMENTROUT, B. 1991. An adaptive model for synchrony in the firefly *pteroptyx malaccae. Journal of Mathematical Biology 29,* 571–585.

ERNST, U., PAWELZIK, K., AND GEISEL, T. 1995. Synchronization induced by temporal delays in pulse-coupled oscillators. *Phys. Rev. Let. 74,* 9, 1570–1573.

ERNST, U., PAWELZIK, K., AND GEISEL, T. 1998. Delay-induced multistable synchronization of biological oscillators. *Phys. Rev. E 57,* 2, 2150–2162.

HONG, Y.-W. AND SCAGLIONE, A. 2005. A scalable synchronization protocol for large scale sensor networks and its applications. *IEEE J. Sel. Areas Commun. 23,* 5, 1085–1099.

JAHNKE, S., MEMMESHEIMER, R., AND TIMME, M. 2008. Stable irregular dynamics in complex neural networks. *Phys. Rev. Let. 100,* 4, 048102.

JAHNKE, S., MEMMESHEIMER, R., AND TIMME, M. 2009. How chaotic is the balanced state? *Frontiers Comp. Neuroscience 10,* Article 13.

JIN, D. Z. 2002. Fast convergence of spike sequences to periodic patterns in recurrent networks. *Phys. Rev. Let. 89,* 20, 208102.

KLINGLMAYR, J., BETTSTETTER, C., AND TIMME, M. 2009. Globally stable synchronization by inhibitory pulse coupling. In *Proc. Intern. Symp. Appl. Sci. Biomed. Comm. Tech. (ISABEL)*. IEEE, Bratislava, Slovak Republic.

KURAMOTO, Y. 1975. Self-entrainment of a population of coupled non-linear oscillators. In *Proc. Intern. Symp. Mathem. Problems Theor. Phys.* Lecture Notes in Physics Series, vol. 39. Springer, 420–422.

KURAMOTO, Y. 1991. Collective synchronization of pulse-coupled oscillators and excitable units. *Physica D 50,* 1, 15–30.

LEIDENFROST, R. AND ELMENREICH, W. 2009. Firefly clock synchronization in an 802.15.4 wireless network. *EURASIP J. Embedded Sys.*, 186406.

LEIDENFROST, R., ELMENREICH, W., AND BETTSTETTER, C. 2010. Fault-tolerant averaging for self-organizing synchronization in wireless ad hoc networks. In *Proc. Intern. Symp. Wireless Commun. Sys. (ISWCS)*. IEEE, York, UK.

LUCARELLI, D. AND WANG, I.-J. 2004. Decentralized synchronization protocols with nearest neighbor communication. In *Proc. ACM Conf. Embedded Networked Sensor Sys. (SenSys)*. Baltimore, MD, USA.

MATHAR, R. AND MATTFELDT, J. 1996. Pulse-coupled decentral synchronization. *SIAM J. Appl. Math. 56,* 4, 1094–1106.

MIROLLO, R. E. AND STROGATZ, S. H. 1990. Synchronization of pulse-coupled biological oscillators. *SIAM J. Appl. Math. 50,* 6, 1645–1662.

PAGLIARI, R. AND SCAGLIONE, A. 2011. Scalable network synchronization with pulse-coupled oscillators. *IEEE Trans. Mobile Comput. 10,* 3, 392–405.

PESKIN, C. S. 1975. *Mathematical Aspects of Heart Physiology*. Courant Institute Lecture Notes. Courant Institute of Mathematical Sciences.

RHOUMA, M. B. H. AND FRIGUI, H. 2001. Self-organization of pulse-coupled oscillators with application to clustering. *IEEE Trans. Pattern Anal. Mach. Intell. 23,* 2, 180–195.

STROGATZ, S. 2003. *SYNC: The emerging science of spontaneous order*. Hyperion books.

TIMME, M. 2002. Collective dynamics in networks of pulse-coupled oscillators. Ph.D. thesis, Georg-August-University Göttingen.

TIMME, M. AND WOLF, F. 2008. The simplest problem in the collective dynamics of neural networks: Is synchrony stable? *Nonlinearity 21,* 7, 1579–1599.

TIMME, M., WOLF, F., AND GEISEL, T. 2002. Coexistence of regular and irregular dynamics in complex networks of pulse-coupled oscillators. *Phys. Rev. Let. 89,* 25, 258701.

TSODYKS, M., I.MITKOV, AND SOMPOLINSKY, H. 1993. Pattern of synchrony in inhomogeneous networks of oscillators with pulse interactions. *Phys. Rev. Let. 71,* 8, 1280–1283.

TYRRELL, A. AND AUER, G. 2008. Decentralized inter-base station synchronization inspired from nature. In *Proc. IEEE Veh. Techn. Conf.* Calgary, Canada, 1–5.

TYRRELL, A., AUER, G., AND BETTSTETTER, C. 2010. Emergent slot synchronization in wireless networks. *IEEE Trans. Mobile Comput. 9,* 5, 719–732.

TYRRELL, A., AUER, G., BETTSTETTER, C., AND NARIPELLA, R. 2010. How does a faulty node disturb decentralized slot synchronization over wireless networks? In *Proc. IEEE Intern. Conf. on Communications (ICC)*. Cape Town, South Africa.

VAN VREESWIJK, C. A., ABBOTT, L. F., AND ERMENTROUT, G. B. 1994. When inhibition, not excitation, synchronizes neural firing. *J. Comp. Neurosci. 1,* 4, 313–321.

WERNER-ALLEN, G., TEWARI, G., PATEL, A., WELSH, M., AND NAGPAL, R. 2005. Firefly-inspired sensor network synchronicity with realistic radio effects. In *Proc. ACM Conf. Embedded Networked Sensor Systems (SenSys)*. San Diego, CA, USA, 142–153.

WINFREE, A. T. 1967. Biological rhythms and the behavior of populations of coupled oscillators. *J. Theoretical Biology 16,* 1, 15–42.