Globally Stable Synchronization by Inhibitory Pulse Coupling

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***Abstract*—We analyze global synchronization of oscillator networks with inhibitory delayed and non-delayed pulse coupling. For a fully-coupled system without delays, we propose simple modifications to the state-of-the-art. We prove that the resulting system converges to full in-phase synchronization of all oscillators from arbitrary initial conditions. For systems with arbitrarily distributed delays we propose a second, related approach. We prove that here a state close to synchrony is globally and asymptotically stable, such that all oscillators lock up to a certain precision from arbitrary initial conditions. The precision of synchronization is determined by the coupling strength and the maximal delay appearing.**

***Index Terms*—Synchronization, networked systems, pulse coupled oscillators, inhibitory coupling, synchronous behavior.**

I. Introduction and Motivation

Time synchronization is a field of interest in many areas, including biomedical and communication systems (see, e.g., [1]–[3]). A common approach to achieve synchrony in a distributed manner has been inspired by the biological phenomenon of synchronously flashing fireflies [4], which can be mathematically described by the theory of pulse coupled oscillators (PCOs) (see [5], [6]). Such models are used in various fields of science. Examples include synchronous behavior in neural networks (see, e.g., [7]) and synchronization in wireless communication networks (see, e.g., [8]–[10]).

Mirollo and Strogatz [6] presented a PCO synchronization model which basically works as follows: Each entity in the system is modeled by an oscillator. Each oscillator maintains a phase ϕ, which uniformly evolves over time t from 0 to 1, resets back to 0, evolves again, and so on. Whenever $\phi = 1$, the oscillator emits a pulse (firing event). If an oscillator receives a pulse from another oscillator, it instantaneously performs a "phase jump" to a higher phase position, where the size of the phase jump depends on its current phase. Given a firing event at time t, the new phase is $\phi(t^+) = \min(\phi(t) + \Delta\phi(t), 1)$, where t^+ represents a time instant that is infinitely shortly after time t. Assuming no delays in the system, this model leads to synchrony of all oscillators for almost all initial phases (global synchronization). Ernst, Pawalzik, and Geisel [11] have shown that in the presence of delays this system does not lead to global synchrony of all oscillators but rather exhibits several coexisting clusters. Werner-Allen and colleagues [9] adapted the model to work in systems with delays and proved the convergence to synchrony of their approach.

All these well-investigated models employ *positive* phase jumps ($\Delta\phi(t) \geq 0$, "excitatory coupling"). Ernst and colleagues [11] suggested to employ *negative* phase jumps ($\Delta\phi(t) \leq 0$, "inhibitory coupling"), as they are relevant in neuroscience modeling and can have certain positive effects on the synchronization process. Such negative coupling was analyzed in more detail by van Vreeswijk *et al.* [12] and Timme *et al.* [13]. In particular Timme *et al.* [13] analyzed the collective network dynamics for arbitrary network topologies and for initial phases close to global synchrony. Still, conclusions for arbitrary initial conditions could not be drawn.

This paper presents inhibitory coupling approaches that lead to synchronization of all oscillators, where synchrony is always guaranteed, independent of the initial conditions, in both systems with and without delay. The key idea is that a firing oscillator omits the phase reset upon firing but just performs a negative phase jump due to self-coupling.

II. Basics on Pulse Coupled Oscillators

Given is a set of identical oscillators. All oscillators are fully-coupled, i.e., every oscillator can receive pulses from every other oscillator. We are using the model description of [6]. For convenience of notation, the time t is dimensionless.

A. Excitatory Coupling

The phase of an oscillator increases linearly over time, i.e., we have

$$\frac{d\phi}{dt} = 1 \tag{1}$$

at all non-event times, with $\phi \in [0, 1]$ and periodic boundary conditions. An oscillator emits a pulse whenever $\phi = 1$ is reached from below.

Upon reception of a such a pulse from another oscillator at time t, each oscillator adjusts its phase from $\phi(t)$ to $\phi(t^+)$. This phase adjustment is determined by the coupling strategy. The new phase $\phi(t^+)$ of the receiving oscillator is

$$\phi(t^+) = \min\left[H(\phi(t)), 1\right], \tag{2}$$

where the transfer function $H(\cdot)$ is given by

$$H(\phi) = U^{-1}\big(U(\phi) + \varepsilon\big). \tag{3}$$

Here, $U(\cdot)$ is an arbitrary twice continuously differentiable function with the following properties: $U(0) = 0$, $U(1) = 1$, $U'(\cdot) > 0$, and $U''(\cdot) < 0$. The term ε is positive for excitatory coupling. The shape of $U(\cdot)$ and the size of ε determine the size of the phase jump. This model leads to synchronous phases of all oscillators independent of the initial phases but a set of measure zero [6].

In case of delays between the sending of a pulse from one oscillator until its reception at another oscillator, this model is no longer capable of providing synchronization (see, e.g., [11]). This is due to the fact that oscillators may receive "echoes" of their own pulses. This phenomenon drives almost synchronous excitatory-coupled oscillators further apart and prohibits the overall system to reach synchrony. The negative effects of delays can be mitigated by introducing a refractory period [14]. It is a time period after each firing event during which a node will not react to received pulses. Introducing a refractory period, synchrony is regained up to a precision determined by the delay.

B. Inhibitory Coupling

A system with negative ε is called an inhibitory coupled system. Using the model description of [13], [15], [16] (here: $\phi \in [-\infty, 1]$), an echo would not drive the oscillators apart. This is a positive effect. The system will in general converge toward a state in which several groups of oscillators exist, but these groups experience phase differences. It is shown that the groups of synchronous oscillators are asymptotically stable. These groups are thus regained if "small disturbances" are injected. The major disadvantage is that the overall system can in general not reach synchrony from arbitrary, global initial conditions [16].

III. Modified Inhibitory Delay-Free Coupling

This section presents and analyzes a modified inhibitory coupling scheme for delay-free systems that reaches full synchrony (zero phase lag) from all initial conditions.

A. Model Description

We follow the basic pulse coupled oscillator model described above with $\phi \in [0, 1]$ and expression (1). Due to inhibitory coupling, the phase adjustment can be simplified to

$$\phi(t^+) = H(\phi(t)) \ . \tag{4}$$

Instead of defining a specific U-function we directly define a phase adjustment function $H(\phi(t))$. Since the results in [6] will not be used for stability analysis, the assumptions from above do not have to be fulfilled. Following the approach of Werner-Allen *et al.* [9], we define a simple linear transfer function

$$H(\phi(t)) = (1 + \alpha) \cdot \phi(t) \ . \tag{5}$$

that mediates the phase adjustment due to interactions. Let α be negative ($-1 < \alpha < 0$), which also guarantees that $\phi(t) \in [0, 1]$ at all times t.

The key idea is as follows: An oscillator increases its phase until threshold and fires. After firing it does not reset its phase to zero, but instead adjusts its phase according to (5) as any other oscillator (instantaneous self-coupling). In other words, all oscillators simply follow the same one reaction upon a firing event. The rules that a pulse coupled oscillator has to follow are summarized in Algorithm 1.

Algorithm 1 Inhibitory delay-free coupling

1) The PCO has its own phase ϕ from 0 to 1.
2) Whenever $\phi(t) = 1$, the PCO sends a pulse and adjusts its phase to $\phi(t^+) = \phi(t) + \Delta\phi(t)$ (instantaneous self-coupling).
3) When receiving a pulse at time t, all other PCOs adjust their phases to $\phi(t^+) = \phi(t) + \Delta\phi(t)$.
4) The phase jump is $\Delta\phi(t) = \alpha \cdot \phi(t)$ with $\alpha < 0$.

An example phase diagram is shown in Figure 1. An oscillator starts at $\phi = 0$ and fires at $\phi = 1$. With our adaptation it then starts again at $\phi = 1 - |\alpha|$, shortening its cycle. From now on the cycle length is $|\alpha|$.

Note the following characteristics. The oscillator that fires first is the one that is closest to the threshold at the starting time. The firing oscillator adjusts its phase, but the order of all phases does not change. Therefore, the firing oscillator will always "stay in the lead," and it is only this oscillator that will perform firing events. This behavior is completely different from the dynamics studied originally by Mirollo and Strogatz, Ernst *et al.*, and Timme *et al.*, where the oscillators closest to the threshold change and all oscillators may perform firing events.

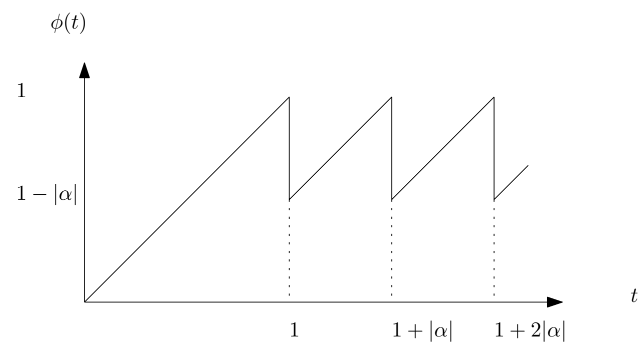

Fig. 1. Evolution of phase $\phi(t)$, illustrating old and new cycle length.

For further analysis, we introduce an operator $f(\cdot)$ expressing the phase adjustment (5):

$$f(\phi(t)) := (1 + \alpha) \cdot \phi(t) \ . \tag{6}$$

The reaction to a single firing can be modeled by applying $f(\cdot)$ to $\phi(t)$.

B. Synchronization Precision

The set of indices of all oscillators is called $\mathcal{I}$. The terms i and j represent elements of $\mathcal{I}$. The collection of the phases of all oscillators at a given time instant is called $\boldsymbol{\phi}$.

A measure for the precision of synchronization at time t is

$$\Gamma\left(\boldsymbol{\phi}(t)\right) := \max_{i,j} \left[\min(|\phi_i(t) - \phi_j(t)|, 1 - |\phi_i(t) - \phi_j(t)|)\right] \ . \tag{7}$$

A synchronized delay-free system has $\Gamma(\boldsymbol{\phi}) = 0$, i.e., all oscillators have the same phase.

Without any firing event, a given precision Γ does not change for all times t. The precision only changes when firing events occur. Thus, to analyze the evolution of the precision, it is sufficient to regard the impact of firings on the precision. In other words, we do not have to consider the precision after time duration t but the precision after n firing events.

For further analysis, we use the inequality

$$\Gamma\left(\boldsymbol{\phi}\right) \leq \left|\max_i \phi_i - \min_i \phi_i\right| \tag{8}$$

and define the norm

$$||\boldsymbol{\phi}|| := \left|\max_i \phi_i - \min_i \phi_i\right| \,. \tag{9}$$

C. Asymptotic Behavior

To show the asymptotic convergence to synchrony, it is sufficient to show that (9) tends to zero. After a single firing event, the phase difference between two given oscillators i and j is

$$|f(\phi_i) - f(\phi_j)| = \left|(1+\alpha)\phi_i - (1+\alpha)\phi_j\right| = (1+\alpha)\,|\phi_i - \phi_j| \,. \tag{10}$$

For n consecutive firing events, we apply the $f(\cdot)$-operator n times, such that the phase difference yields

$$|f^n(\phi_i) - f^n(\phi_j)| = (1+\alpha)^n\,|\phi_i - \phi_j|. \tag{11}$$

To obtain the largest phase difference in the system, we choose the maximum ϕ_i and the minimum ϕ_j, which yields, by using (9) and the fact that $\max(\cdot)$ and $\min(\cdot)$ are linear operators, the expression

$$||f^n(\boldsymbol{\phi})|| = (1+\alpha)^n\,||\boldsymbol{\phi}|| \,. \tag{12}$$

Finally, we take the limit for n and get

$$\lim_{n\to\infty} ||f^n(\boldsymbol{\phi})|| = \lim_{n\to\infty} (1+\alpha)^n ||\boldsymbol{\phi}|| = 0 \tag{13}$$

because $\alpha < 0$ and the term $||\boldsymbol{\phi}||$ is determined by the starting configuration and is thus constant. In summary, the convergence of the system toward zero phase lag is guaranteed.

IV. Modified Inhibitory Delayed Coupling

This section adapts the inhibitory coupling scheme for a system with delays, such that it synchronizes up to a phase lag caused by the appearing delay.

A. Model Description

Let τ_i represent the delay between a firing oscillator and a receiving oscillator i. We assume that the delays τ_i are independent from each other and in general different. The maximum possible delay is denoted by $\tau_{\max} = \max_i \tau_i$. Upon the firing of an oscillator i, the oscillators behave as follows:

$$\phi_i(t) = 1 \Rightarrow \begin{cases} \phi_i(t^+) = (1+\alpha)\cdot\phi_i(t) \\ \phi_j(t^+ + \tau_j) = (1+\alpha)\cdot\phi_j(t+\tau_j)\,;\; j \neq i \end{cases} . \tag{14}$$

The delay causes the system to react to a firing event only after a time τ. If multiple oscillators are in the lead and fire at the same time t, the pulses will in general have different delays to a given receiving oscillator. The oscillator thus receives multiple pulses. We want to mitigate this delay spread, such that a receiving oscillator adjusts its phase only once. For this reason, we introduce a refractory period after each phase adjustment. During this refractory period, an oscillator does not react to firing pulses. Hence, a receiving oscillator will just consider the first received pulse and ignore the other pulses. Taking into account the time spread of the pulse emissions at different oscillators as well as the delay spread to the receiving oscillator, we set the duration of the refractory period to $2\tau_{\max}$. In summary, an oscillator that just adjusted its phase has to pass the phase instance $1 - |\alpha| + 2\tau_{\max}$ before being able to adjust again. The entire scheme is summarized in Algorithm 2.

Algorithm 2 Inhibitory delayed coupling

1) The PCO has its own phase ϕ from 0 to 1.
2) When $\phi(t) = 1$, the PCO fires and adjusts its phase to $\phi(t^+) = \phi(t) + \Delta\phi(t)$ (instantaneous self-coupling).
3) When receiving a firing at time t', the PCO adjusts its phase to $\phi(t'^+) = \phi(t') + \Delta\phi(t')$
4) For $\Delta\phi(t')$ with $\alpha < 0$:
   a) If $\phi(t') > 1 - |\alpha| + 2\tau_{\max}$, set $\Delta\phi(t') = \alpha\cdot\phi(t')$.
   b) If $\phi(t') \leq 1 - |\alpha| + 2\tau_{\max}$, set $\Delta\phi(t') = 0$.

Now we define an operator $g(\cdot)$ representing the delayed phase adjustment:

$$g\left(\phi_i(t)\right) = f\left(\phi_i(t+\tau_i)\right) = (1+\alpha)\cdot\phi_i(t+\tau_i)\,. \tag{15}$$

With the refractory strategy, the operator $g(\cdot)$ will be applied only once for each firing event.

B. Synchronization Precision

Consider the definition of the synchronization precision given in (7). We use $|\alpha|$ as cycle length instead of 1, after the first cycle. Again, the precision will not change over time if no firing events occur. Therefore we only have to consider firing events. Using the refractory strategy, the application of the operator $g(\cdot)$ "communicates" a firing event only once to a receiving oscillator. As a result, we only have to consider the iterated application of $g(\cdot)$ on the evolution of the precision. We employ again the norm $||\boldsymbol{\phi}||$ given in (9) as an upper bound for the achieved precision, whenever refractory time passed.

C. Asymptotic Behavior

As the phase increases linearly over time, we have $\phi_i(t+\tau_i) \leq \phi_i(t) + \tau_i$. Consequently, (15) is bounded above by

$$g\left(\phi_i(t)\right) \;\leq\; f\left(\phi_i(t)+\tau_i\right) = (1+\alpha)\cdot\left(\phi_i(t)+\tau_i\right), \tag{16}$$

as $f(\cdot)$ is a linear operator. It is bounded below by

$$g\left(\phi_i(t)\right) \;\geq\; f\left(\phi_i(t)\right) = (1+\alpha)\cdot\phi_i(t)\,. \tag{17}$$

After a single firing event, the phase difference between oscillators i and j is bounded above by

$$\left|g(\phi_i) - g(\phi_j)\right| \;\leq\; (1+\alpha)\cdot\left(\left|\phi_i - \phi_j\right| + \tau_i\right)\,. \tag{18}$$

Subsequent firing events are indexed by $l = 1, \ldots, n$. For a fire event l, the delay of a fire pulse at oscillator i is denoted by $\tau_{i,l}$. The recursive application of the operator $g(\cdot)$ yields

$$g^n(\phi_i) \ \le\ (1+\alpha)^n \cdot \phi_i \ +\ \sum_{l=1}^{n}(1+\alpha)^{n-l+1} \cdot \tau_{i,l} \qquad (19)$$

and

$$g^n(\phi_i) \ \ge\ (1+\alpha)^n \cdot \phi_i \,. \qquad (20)$$

For convenience of notation, we define

$$\omega_{i,n} := \sum_{l=1}^{n}(1+\alpha)^{n-l+1}\tau_{i,l} \,. \qquad (21)$$

The phase difference of two oscillators i and j after n firing events is thus bounded by

$$\left|g^n(\phi_i) - g^n(\phi_j)\right| \le \left|(1+\alpha)^n\phi_i + \omega_{i,n} - (1+\alpha)^n\phi_j\right| , \quad (22)$$

where the right hand side simplifies to

$$(1+\alpha)^n \, |\phi_i - \phi_j| + \omega_{i,n} \,. \qquad (23)$$

To obtain the largest phase difference in the system, we choose the maximum ϕ_i and the minimum ϕ_j, which yields

$$||g^n(\boldsymbol{\phi})|| \ \le\ (1+\alpha)^n \, ||\boldsymbol{\phi}|| + \tau_{\max} \sum_{l=1}^{n}(1+\alpha)^{n-l+1} \,. \quad (24)$$

Taking the limit for n, the infinite sum converges, and we get

$$\lim_{n\to\infty} ||g^n(\boldsymbol{\phi})|| \ \le\ \tau_{\max}\left(\frac{1}{|\alpha|} - 1\right) . \qquad (25)$$

This shows that, starting from an arbitrary phase constellation, synchronization is guaranteed up to a phase spread of at most $\Theta := \tau_{\max}\left(\frac{1}{|\alpha|} - 1\right)$ with $0 < |\alpha| < 1$.

V. Conclusions

Simple modifications to known pulse coupled oscillator models with inhibitory couplings imply that full synchrony is reached from arbitrary initial conditions. For a delay-free system, we have shown analytically that it reaches global synchrony with zero phase lag. For a system with delays, we demonstrated analytically that a close-to-synchronous state is reached (in finite time) as well. In this close-to-synchronous state, all phases are the same up to a precision of at most Θ, where this bound is determined by the maximal delay in the system and the coupling strength. These global stability results of synchrony complement results on local synchronization as found, e.g., in [16].

We remark that the rise function $U(\phi) = \frac{1}{b}\ln(1+(e^b-1)\phi)$ yields an affine transfer function $H(\phi) = A + B\phi$ as in [17], such that the effect on the phase *differences* is identical to that presented in the above analysis for systems without delays. In the presence of distributed delays, the consequences of this feature need to be studied in the future.

Last but not least, the presented coupling approach may be employed in more advanced synchronization algorithms such as [10]. It might thus be possible to give analytical results for such algorithms as well. Results on this topic may help ensuring synchronization in distributed wireless communication networks.

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