Failure-Resilient Ad Hoc and Sensor Networks in a Shadow Fading Environment

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Abstract

This paper addresses the topological design of wireless multihop networks that are robust against node failures. Given is the following scenario: The nodes are randomly distributed according to a homogeneous Poisson point process of density ρ, each node has the same transmission capabilities, and the wireless channel suffers from a log–normal shadow fading. We investigate the minimum node density ρ required to ensure that all nodes inside a randomly chosen area of size A are k–connected with high probability ($k \in \mathbb{N}$). We derive a tight lower bound for this node density as a function of the channel parameters and compute it for a number of scenarios. The results give insight into how fading affects the network topology.

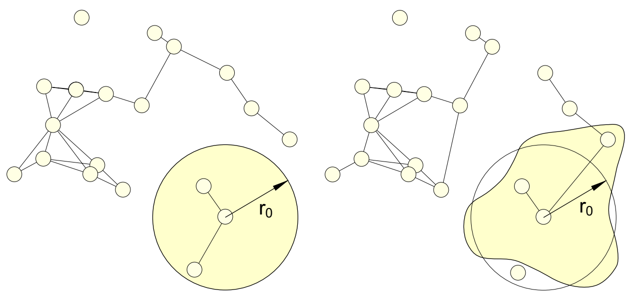
Keywords*: Ad hoc networks, sensor networks, multihop, k-connectivity, robustness, failure-resilience, shadowing.*

1. Introduction

A fundamental property of each communication network is its connectivity. In cellular systems, it is the operator's responsibility to enable communication among users. It deploys a sufficient number of base stations and maintains the access and core networks. This network infrastructure guarantees, to a certain, extend the connectivity among mobile users. In contrast to this, the connectivity in wireless multihop networks (i.e., ad hoc and sensor networks) cannot be guaranteed. Each node acts as a relay for other nodes, thus the level of connectivity depends on the spatial density and transmission characteristics of the mobile nodes themselves. As the location and mobility of the nodes are non–deterministic, we can only give probabilistic measures for the connectivity. This stochastic aspect raises a number of interesting problems, which have been addressed by several researchers.

Cheng and Robertazzi [1] investigate how far a node's broadcast message percolates, assuming that nodes are randomly distributed according to a homogeneous Poisson point process on an infinitely large area. Piret [2], Gupta and Kumar [3], Santi and Blough [4, 5], Bettstetter [6–8], and Desai [9] study the connectivity of a given number of nodes that are placed on a finite area using a uniform random distribution. They analyze how high the transmission power of the nodes must be such that the network is connected, i.e., there is a communication path between each pair of nodes. Non–uniform random distributions are addressed in [8]. Dousse *et al.* [10] consider Poisson distributed nodes on an infinite line and give an expression for the probability that two nodes with a given distance can establish a multihop path between them.

All these papers have one assumption in common: they describe the wireless link between nodes with a *very simple channel model*. In this model, two nodes are linked together, if they are not further apart than a certain threshold distance, the so–called transmission range r_0 (see Fig. 1 a). The resulting disk graph model is convenient for analytical computations and has been a good starting point to understand the nature of multihop connectivity. We have to admit, however, that radio links look very different in reality. In particular, the transmission range is not rotationally symmetric due to shadowing effects and inhomogeneities of the antenna (see Fig. 1 b).



(a) Purely geometric model (b) Shadowing fading model

Figure 1. Illustration of link models

A sound analytical study of the connectivity (and other topology properties) with a more realistic channel model is still an open issue. Just recently, first steps to obtain a basic understanding have been made in [11] and [12]. Our paper [11] studied the following problem: *Given a shadow fading environment, what is the minimum density (nodes per unit area) of Poisson distributed nodes to achieve a fully connected network in a subarea of given size.* The paper leads to the initially surprising result that a higher fading variance may help the network to become connected, i.e., the higher the fading variance, the lower is the node density required to achieve a connected network. A similar result has been observed qualitatively in [12] by studying the existence of an infinite connected component. Here, the authors conclude that anisotropic radio coverage allows an infinite connected component to appear at a lower density than perfect circular coverage does.

The paper at hand extends previous analysis of the above problem with the following two main contributions:

- Previous work [11] used multi–dimensional numerical integrals to compute a tight bound for the critical node density required for connectivity. We now give a solution that requires no integrals at all. This allows us to perform a *more thorough analysis*, since the computational complexity is much lower now.

- We study a network design that is robust against node failures, i.e., we require that the network is not only connected but *k–connected* ($k \in \mathbb{N}$), which means that $(k-1)$ nodes may fail and the network is still guaranteed to be still connected. This issue was already addressed in [6, 13] but only for the simple geometric link model.

The remainder of this paper is organized as follows: Section 2 describes the used network model, including the model for shadowing. Section 3, addresses the level of connectivity from the viewpoint of a node. It gives an equation for the probability density function of the number of neighbors of a node. Section 4 computes a very tight bound for the minimum node density ρ required to achieve a k–connected network in a given subarea. Several plots illustrate this density as a function of the link parameters. Finally, Section 5 concludes and gives ideas for future research.

2. Network Model

2.1. Spatial Node Distribution

The spatial distribution of the nodes is described by a random point process on an infinite plane. We use a *homogeneous Poisson process* of density ρ (see Appendix). It can be regarded as the limiting form of a uniform distribution of n nodes on an area of size A, as n and $A \to \infty$ while the ratio $\rho = n/A$ remains constant.

2.2. Wireless Link Model

We now define which nodes establish a wireless link between each other. Let us consider a node that transmits a signal with power p_t and another node that receives this signal with power p_r. The signal is received properly if p_r is larger than or equal to a certain threshold power $p_{r,th}$, which is denoted as *receiver sensitivity*. We say that the sender establishes a *wireless link* to the receiver if $p_r \geq p_{r,th}$. In the following, we assume that all nodes have the same p_t and $p_{r,th}$. Thus, all links can be considered as being undirected (bidirectional).

The *signal attenuation* from the sender to the receiver is defined by $\beta = \frac{p_t}{p_r}$; it can be expressed in terms of decibel as

$$\beta = 10 \log_{10}\left(\frac{p_t}{p_r}\right) \text{ dB}\,. \tag{1}$$

For given p_t and $p_{r,th}$, two nodes can can communicate via a direct link (i.e., they are *neighbors*), if the attenuation between them fulfills $\beta \leq \beta_{th}$, with the threshold attenuation

$$\beta_{th} = 10 \log_{10}\left(\frac{p_t}{p_{r,th}}\right) \text{ dB}\,. \tag{2}$$

We describe the attenuation β as a consequence of two characteristics of the wireless channel.

2.2.1. Path loss caused by distance

A simple model to describe the wireless channel is to assume that the received signal value p_r falls off proportional to some power α of the distance s from the sending node, i.e.,

$$p_r = \left(\frac{s}{1\,\text{m}}\right)^{-\alpha} p_t\,. \tag{3}$$

The term α denotes the *path loss exponent* of the environment, which is typically ranging between 2 and 5. For example, we have $\alpha \approx 2$ in free space and $\alpha \approx 3$ in an urban outdoor environment [14]. With this model, the attenuation is

$$\beta_0 = \alpha\, 10 \log_{10}\left(\frac{s}{1\,\text{m}}\right) \text{ dB}\,. \tag{4}$$

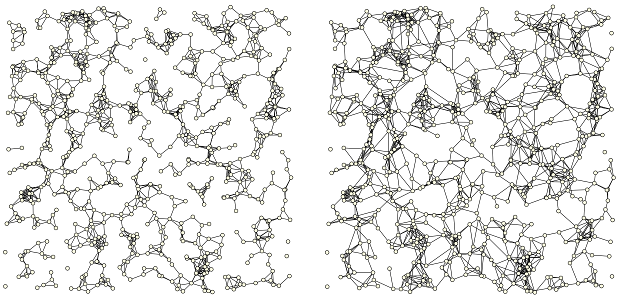
Using omnidirectional antennas, a node has links to all nodes that are currently located within a circle of radius

$$r_0 = \left(\frac{p_t}{p_{r,th}}\right)^{\frac{1}{\alpha}} \text{ m}\,. \tag{5}$$

around its position (see Fig. 1 a and 2 a).

2.2.2. Shadow fading

In an environment that contains objects — e.g., buildings, cars, walls, furniture — the distance s is no longer



(a) Purely geometric link model ($\beta_{th} = 69\,\text{dB}$, $\alpha = 3$, $\sigma = 0\,\text{dB}$)

(b) Shadow fading link model ($\beta_{th} = 69\,\text{dB}$, $\alpha = 3$, $\sigma = 4\,\text{dB}$)

Figure 2. Network with $\rho = 4.375 \cdot 10^{-5}\,\text{m}^{-2}$ on $4000 \times 4000\,\text{m}^2$ with $r_0 = 200\,\text{m}$

sufficient to determine the attenuation. This is because different objects "shadow" the signal in different ways. Thus, receivers located at the same distance s from the sender (but at different absolute positions) may experience different values for p_r. Since the properties of the objects (e.g., size, location) are in general unknown, stochastic models for shadowing are used. Measurements have shown that the received power p_r can be approximated by a log–normal probability density around a mean given by (3). Converting this log–normal probability density to dB, yields a normal probability density. The variation of the attenuation around β_0 is thus described by the random variable β_s with

$$f_{\beta_s}(\beta_s) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\beta_s^2}{2\sigma^2}\right). \tag{6}$$

The standard deviation σ is given in dB; typical values are up to 10 dB [15].

Combining path loss and shadowing, the overall attenuation is given by

$$\beta = \beta_0 + \beta_s \quad \text{in dB}, \tag{7}$$

where β_0 is a geometric, purely deterministic component and β_s a purely random component. Figure 2 b illustrates an example topology. The area covered by a node is no longer a circle but could look like in Figure 1 b. It is now possible that: (i) there may be a link between two nodes that are located more than a distance of r_0 away from each other; (ii) there may be no link between two nodes that are located less than r_0 away from each other.

3. Number of Neighbors of a Node

This section studies the level of connectivity from the local viewpoint of a node. We consider the number of neighbors of a node, which is called its *degree* $d \in \mathbb{N}_0$ (random variable D). Given the above modeling assumptions, a node's degree is Poisson distributed, i.e.,

$$P(D = d) = \frac{\mu^d}{d!} e^{-\mu}. \tag{8}$$

The expected node degree can be expressed as

$$\mu = E\{D\} = \rho\pi r^2 \tag{9}$$

where (adapted from [16][1])

$$r = 10^{\eta} e^{\zeta}\,\text{m} \tag{10}$$

with

$$\eta = \frac{\beta_{th}/\text{dB}}{10\alpha} \quad \text{and} \quad \zeta = \left(\frac{\ln(10)\,\sigma/\text{dB}}{10\alpha}\right)^2. \tag{11}$$

The expression $\ln(\cdot)$ denotes the natural logarithm. The term r is determined by the transmission characteristics of the nodes and the channel environment; it can be interpreted as the *effective transmission range*. If there is no shadow fading, i.e., $\sigma = 0$, we obtain with (2) and (5) the well–known expression $\mu = \rho\pi r_0^2$. If σ increases, the expected degree increases as well. In other words, a higher fading variance leads to a higher level of connectivity. This phenomenon occurs because a node looses links to (some) nodes that are located closer than r_0, but in turn establishes new links to (more) nodes that are located further away than r_0.

The probability that a node has at most d neighbors is

$$P(D \le d) = e^{-\mu} \sum_{D=0}^{d} \frac{\mu^D}{D!} = \frac{\Gamma(d+1, \mu)}{d!}, \tag{12}$$

where $\Gamma(a,b)$ is the incomplete Gamma function $\Gamma(a,b) = (a-1)!\, e^{-b} \sum_{i=0}^{a-1} \frac{b^i}{i!}$, with $a \in \mathbb{N}$. The probability that a node is isolated is $P(D = 0) = \Gamma(1, \mu) = e^{-\mu}$.

4. The k–Connectivity of a Subarea

4.1. Problem Statement

We now consider a finite subarea $\mathbf{A}$ of the system plane. It physical size is denoted by $A = \|\mathbf{A}\|$. The nodes in $\mathbf{A}$ are said to be *k–connected* ($k \in \mathbb{N}$) if for each node pair there exist at least k node–disjoint paths connecting them. Equivalently, if any $(k-1)$ nodes fail, the nodes are guaranteed to be still (1–)connected. Nodes outside of $\mathbf{A}$ may act as relay nodes to connect nodes inside of $\mathbf{A}$.

We are interested in the probability $P(k\text{–con})$ that all nodes within $\mathbf{A}$ are k–connected. More specifically, we

[1] The paper [16] uses a slightly different channel model. To obtain (10) and (11) from the results of Section IV in [16], we have to substitute $k_0 = 0$ and $k_1 \ln(R) = 10\alpha \log_{10}(s)$.

would like to know the minimum node density ρ that is required to achieve a high probability $P(k{-}\mathrm{con}) = 99\,\%$. In the following, we first derive a lower bound for this density and then show that this bound is very tight in most practical scenarios.

4.2. A Lower Bound

The event that each node in **A** has at least k neighbors, i.e., the event that the *minimum node degree* $D_{min} = \min_{\forall \mathrm{node}} D(\mathrm{node})$ of the random network in **A** is at least k, is a necessary but not sufficient condition for the event that all nodes in **A** are k–connected. Hence, for given A, the critical node density required to achieve a network in which $D_{min} \geq k$ is guaranteed with a probability p is a lower bound for the critical node density required to achieve a k–connected network with the same probability p. In mathematical notation,

$$\rho\,(P(D_{min} \geq k) = p) \;\leq\; \rho\,(P(k{-}\mathrm{con}) = p)\;. \qquad (13)$$

Let us now compute this bound. The random variable N represents the number of nodes in **A**. It is Poisson distributed per definition with mean $\lambda = E\{N\} = \rho A$. In the following, we require that many nodes are located in **A**, say $\lambda > 100$. Moreover, we know that each node must have a very low probability $P(D<k)$ to achieve $P(D_{min} \geq k)$ close to one. Given these two assumptions we can state that the event (12) can be considered to be "almost independent" from node to node.

If we assume for a moment that the number of nodes in **A** is known (i.e., $N = n$), we can write

$$\begin{aligned} &P(D_{min} \geq k \mid N = n) \approx \\ &\quad\approx\;\; P(D \geq k)^n = \big(1 - P(D \leq k-1)\big)^n . \end{aligned} \qquad (14)$$

It denotes the probability that each one of the n nodes in **A** has at least k neighbors under the condition that exactly n nodes are located in **A**. The unconditional probability $P(D_{min} \geq k)$ is given by

$$P(D_{min} \geq k) = \sum_{n=1}^{\infty} P(D_{min} \geq k \mid N{=}n)\,P(N{=}n)\;. \qquad (15)$$

Simplification of this sum yields

$$\begin{aligned} P(D_{min} \geq k) \;&\approx\; \exp\Big(-\rho A\, P(D \leq k-1)\Big) \\ &=\; \exp\left(-\rho A\,\frac{\Gamma(k,\rho\pi r^2)}{(k-1)!}\right). \end{aligned} \qquad (16)$$

As a final step, we solve $P(D_{min} \geq k) = 99\,\%$ for ρ. The results are shown in Figure 3 for four different area sizes with $k = 1,\ldots,4$. The impact of the shadowing coefficient σ can be observed in Figures 4 and 5. For increasing σ, a lower density is needed to achieve the same connectivity, i.e., a higher σ helps the network to become connected. This result holds because our channel model decouples fading (σ) from pathloss (α). In a real–world environment, a higher σ typically comes with a higher path loss α, which in turn requires a higher density.

4.3. Tightness Of Bound

The question arises as to the tightness of the lower bound $\rho\,(P(D_{min} \geq k) = 99\,\%)$. From previous research, we know that $\rho\,(P(D_{min} \geq k) = p)$ is a very tight bound for $\rho\,(P(k{-}\mathrm{con}) = p)$, if p is close 1, in the following cases:

- if we use a purely geometric channel model ($\sigma = 0$) and require k–connectivity for arbitrary k (see [6, 8], based on [17]),
- if we use a shadow fading environment (arbitrary σ) and require pure 1–connectivity (see [11]).

Furthermore, we have observed in [6,8] that the approximation of $\rho\,(P(k{-}\mathrm{con}) = 99\,\%)$ with $\rho\,(P(D_{min} \geq k) = 99\,\%)$ becomes better for increasing k. These effects can be regarded as threshold effects also known in the theory of (pure) random graphs without any geometric component [18]. Combining these observations implies that $\rho\,(P(D_{min} \geq k) = 99\,\%)$ is also a very tight bound in our scenario, i.e.,

$$\rho\,(P(k{-}\mathrm{con}) = 99\,\%) = \rho\,(P(D_{min} \geq k) = 99\,\%) + \rho_\varepsilon \qquad (17)$$

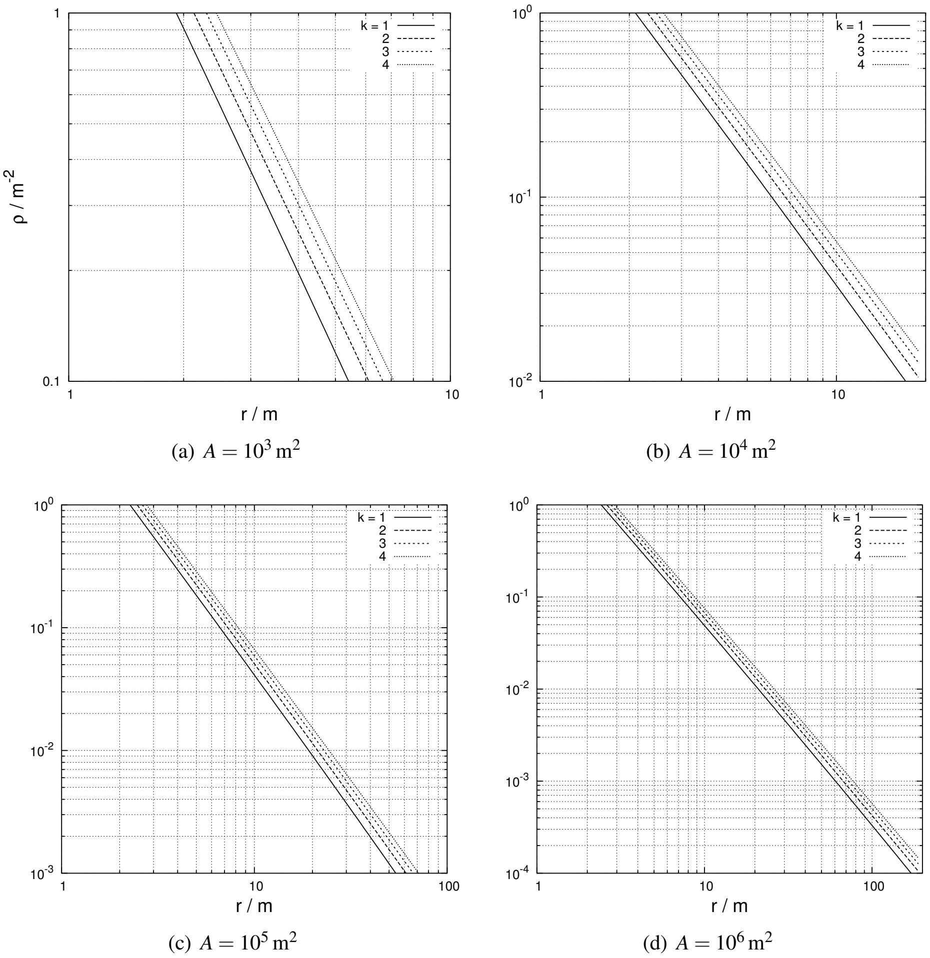
with $\rho_\varepsilon > 0$ and $\rho_\varepsilon / \rho\,(P(D_{min} \geq k) = 99\,\%)$ close to 0.

To verify this conjecture, we perform a number of simulations. For given channel parameters, we tune ρ to achieve $P(k{-}\mathrm{con}) = 99\,\% \pm 0.01\,\%$, based on the statistical average of 10 000 random topologies and a circular **A**. The results are depicted as small circles in Figures 4 and 5; they show a very good match of analytical and simulation–based values.

In conclusion, from an engineering perspective, it is sufficient to compute $\rho\,(P(D_{min} \geq k) = 99\,\%)$ and use it as a very tight approximation for $\rho\,(P(k{-}\mathrm{con}) = 99\,\%)$.

4.4. Example: *k*–connectivity of BTnodes

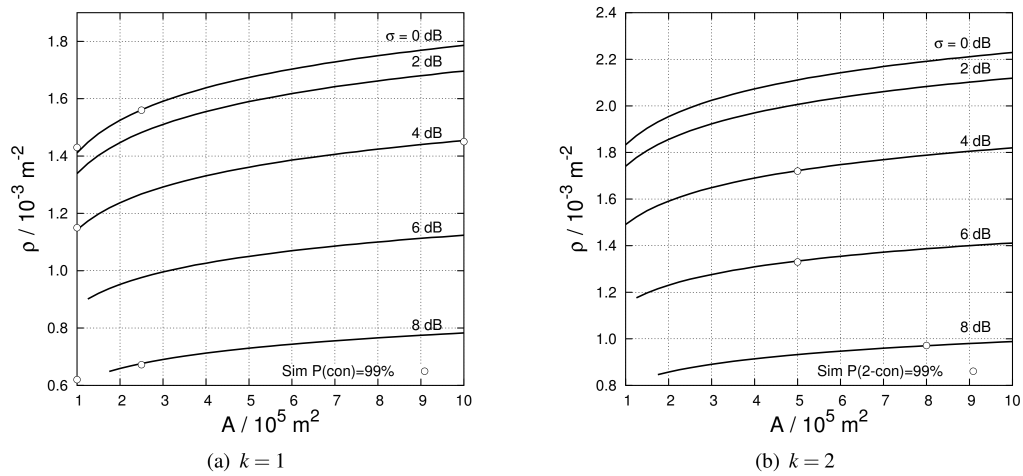
A circular area of size $A = 10^4\,\mathrm{m}^2$ with pathloss $\alpha = 3$ and $\sigma = 4\,\mathrm{dB}$ should be covered with Bluetooth sensors (e.g., BTnodes [19]). What is the required density ρ such that the network is 2–connected? A Bluetooth device has a transmission power $p_t = 100\,\mathrm{mW}$ and a sensitivity $p_r = 100\,\mu\mathrm{W}$; this corresponds to $\beta_{th} = 30\,\mathrm{dB}$. Using (10) and (11), we obtain an effective range of $r = 11.0\,\mathrm{m}$. With Figure 3, the required density is $\rho = 0.035\,\mathrm{m}^{-2}$, i.e., about $n = 350$ devices are needed on the area.



(a) $A = 10^3\,\mathrm{m}^2$ (b) $A = 10^4\,\mathrm{m}^2$

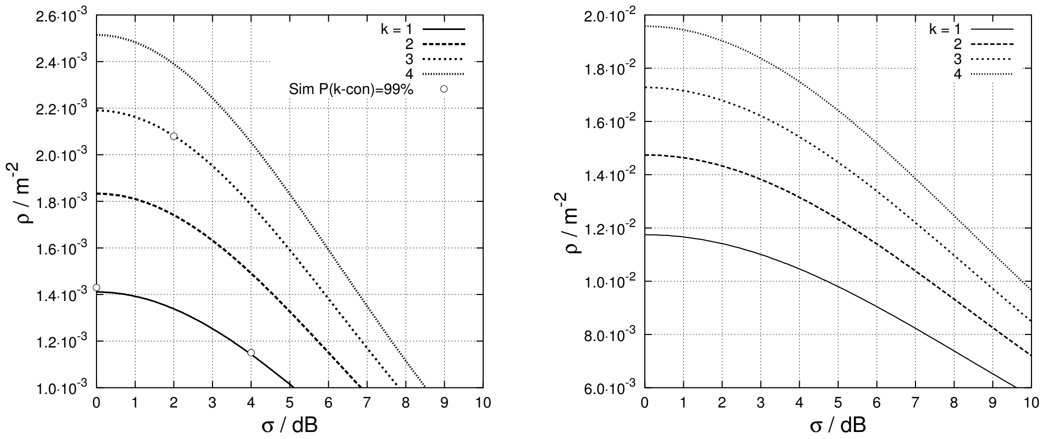
(c) $A = 10^5\,\mathrm{m}^2$ (d) $A = 10^6\,\mathrm{m}^2$

Figure 3. Critical node density ρ resulting in $P(D_{min} \geq k) = 99\,\%$ of nodes inside area A



(a) $k = 1$ (b) $k = 2$

Figure 4. Critical node density ρ resulting in $P(D_{min} \geq k) = 99\,\%$ of nodes inside area A ($\beta_{th} = 50\,$dB, $\alpha = 3$)



(a) $\alpha = 3$ (b) $\alpha = 4$

Figure 5. Critical node density ρ resulting in $P(D_{min} \geq k) = 99\,\%$ of nodes inside area $A = 10^5\,\mathrm{m}^2$ ($\beta_{th} = 50\,\mathrm{dB}$)

5. Conclusions and Further Work

This paper provides guidelines for the topological "planning" of robust wireless ad hoc and sensor networks (e.g., sensor dust applications [20]). Given the channel and node transmission characteristics — which are both completely described by the effective range r, we can compute the node density ρ that ensures, with probability close to one, the k–connectivity of all nodes inside an area of given size.

Knowing that a network is k–connected is especially beneficial if multipath routing [21] is desired, which helps to improve the robustness of packet delivery or/and balances the routing load among the nodes. If full connectivity of the nodes is not a design goal, our results still give a notion of the "level of connectivity" in the network.

We see two main issues for further research in this field. First, the results on the node degree can be applied to message percolation problems, e.g., to determine the critical node density that creates a connected component of infinite size. Second, the impact of interferences (as in [22]) should be studied in combination with shadowing. All in all, the goal of the research community should be to develop a common theory for connectivity and percolation issues in wireless multihop networks.

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Appendix: Homogeneous Poisson Process

A homogeneous Poisson point process is defined by the following two properties [23]:

- The number of nodes N in each finite subarea $\mathbf{A}$ of size $\|\mathbf{A}\| = A$ follows a Poisson distribution, i.e.,

$$P\,(n\,\text{nodes in}\,\mathbf{A}) \;=\; P\,(N=n) \;=\; \frac{\mu^n}{n!}\,e^{-\mu}\,;\, n \in \mathbb{N}_0\,. \tag{18}$$

  with a mean value $E\{N\} = \mu = \rho A$.

- The number of nodes N_i in disjoint (non–overlapping) areas $\mathbf{A}_i$, $i \in \mathbb{N}$, are independent random variables, i.e.,

$$P\,(N_1 = n_1 \wedge N_2 = n_2 \wedge \ldots \wedge N_\kappa = n_\kappa) \;=\; \prod_{i=1}^{\kappa} P\,(N_i = n_i)\,. \tag{19}$$

We denote this process as being homogeneous, if ρ is constant over the entire infinitely large area. In other words, the outcome of the random variable N only depends on the size of the subarea $\mathbf{A}$ but not on its particular location or shape.