On the Connectivity of Wireless Multihop Networks with Homogeneous and Inhomogeneous Range Assignment

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***Abstract*— This paper investigates the connectivity of wireless multihop networks with uniformly randomly distributed nodes. We analyze the required transmission range that creates, for a given node density, an almost surely k–connected topology. Besides scenarios in which each node has the same range, we discuss inhomogeneous range assignments. Our results are of practical value for the task of setting parameters in network–level simulations of ad hoc networks and in the design of wireless sensor networks.**

I. Introduction

Wireless multihop networks are formed by a group of nodes that communicate with each other in a decentralized and self–organizing manner. Each node can act as a router to forward traffic toward its destination. Examples for such networks are mobile ad hoc networks or wireless sensor networks.

This paper investigates a fundamental property of a wireless multihop network: its *connectivity*. Whereas in cellular systems, it is sufficient that each mobile node has a wireless link to at least one base station, the situation in decentralized wireless multihop networks is more sophisticated. To achieve a connected network, a wireless multihop path must exist from each node to each other node. Each single node contributes to the connectivity of the entire network; if a node fails the connectivity might be destroyed. The probability for a network to be connected depends on the density of nodes and their transmission ranges.

Let us assume a typical simulation scenario: a number of n nodes are placed uniformly at random on a square system area of size $A = a \times a$. A simple free–space radio link model is used, in which each node has a given transmission range using an omnidirectional antenna. Two nodes are able to communicate directly via a wireless link, if they are within range of each other. Only bidirectional links are considered.

In this context, we address the following problems. First, we consider the case in which each of the n nodes has the same transmission range r_0 (homogeneous range assignment). We ask: for a given node density $\rho = n/A$, what is the minimum range r_0 to achieve *with high probability* a connected network? In other words, for a given area A, which (r_0, n)–tuples achieve high connectivity? Second, we consider an inhomogeneous range assignment (n_1 nodes with r_1, n_2 nodes with $r_2, \ldots$), and we investigate how the probability of being connected changes in this case. In addition to the basic problem "connected network," we also consider a network design that is robust against failures: how can we achieve a multihop network that is still connected if some nodes fail?

Definitions: The number of neighbors of a node (i.e., its number of links) is denoted by its *degree* d. A node with $d = 0$ is said to be *isolated*. The *minimum node degree* d_{min} is the smallest node degree over all nodes in the network.

A network is said to be *connected*, if for every pair of nodes there exists a path between them, and otherwise it is *disconnected*. A connected network has always $d_{min} > 0$, but the reverse implication is not necessarily true. A network is said to be *k–connected* ($k \in \mathbb{N}$) if for each node pair there exist at least k mutually independent paths connecting them. Equivalently, a network is k–connected if and only if no set of $(k-1)$ nodes exists whose removal would disconnect the network. In the following, the probability that a network is k–connected is denoted as $P(k\text{–con})$. This paper considers $k = 1,\ 2,$ and 3. For $P(1\text{–con})$, we write $P(\text{con})$. We say that a network is *almost surely (a.s.) k–connected*, if $P(k\text{–con}) \geq 0.95$.

Related Work: Early work on connectivity in multihop radio networks has been published in [1], [2], and [3]. More recently, Gupta and Kumar [4], Santi *et al.* [5], Bettstetter [6], and Dousse *et al.* [7] seize this problem again. Each of them takes a quite different approach to modeling and solving the problem. So far, only 1–connectivity and homogeneous range assignment has been considered by other authors.

This paper first deepens our research on k–connectivity with homogeneous range assignment [6], and, second, investigates the impact of inhomogeneous range assignment on connectivity.

II. Homogeneous Range Assignment

In order to investigate the k–connectivity of a wireless multihop network by simulation, we generate a number of random topologies, check the connectivity for each topology, and then take the average, such that $P(k\text{–con})$ is estimated by the percentage of k–connected topologies. Fig. 1 shows the results for $P(k\text{–con})$ over the range r_0 for $n = 500$ and 1000 nodes on an area $A = a \times a = 1000 \times 1000\ \text{m}^2$. To avoid border effects [8][9], we use a toroidal distance metric, i.e., nodes at the border of the area can have wireless links via the borderline to nodes on the opposite side of the area (also see [6]). Starting at

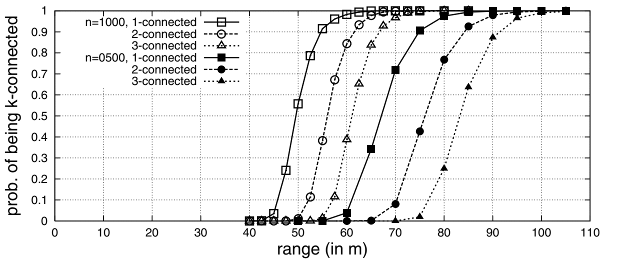


Fig. 1. Connectivity for $n = 1000$ and $n = 500$ nodes on $A = 1000 \times 1000\ \text{m}^2$, toroidal distance, 1000 random topologies

$r_0 = 0$ and increasing r_0, $P(\text{con})$ remains zero until a certain critical range is achieved. For example, $n = 1000$ nodes with $r_0 \leq 45$ m yields an a.s. unconnected network. Once r_0 is larger than this critical value, $P(\text{con})$ increases, until an a.s. connected network is achieved. For example, $n = 1000$ nodes require $r_0 \geq 57$ m to make the network connected with high probability (95%), and for $n = 500$ we must set $r_0 \geq 78$ m. Additional range increments yield 2 and 3–connected topologies.

We are especially interested in the upper threshold ranges $r_0(P(k\text{–con}) \approx 1)$ needed to achieve an a.s. k–connected network. Let us take an approach to calculate these ranges as a function of ρ. From [6], we know the following theorem.

Theorem 1 (On the k–connectivity of homogeneous ad hoc networks): A wireless multihop network with n uniformly distributed nodes ($n \gg 1$, $\rho = n/A$) with a homogeneous range assignment r_0 ($r_0^2\pi \ll A$) achieves

$$P(k\text{–con}) \approxeq P(d_{min} \geq k) \quad \text{for probabilities close to 1,} \tag{1}$$

if n is high enough. If border effects are not present, we have $P(d = i) = \frac{\mu^i}{i!}\, e^{-\mu}$ with $\mu = \rho\pi r_0^2$. Since these probabilities are "almost independent" with the above assumptions, we obtain

$$P(d_{min} \geq k) \;=\; \left(1 - e^{-\mu}\sum_{i=0}^{k-1}\frac{\mu^i}{i!}\right)^n \tag{2}$$

$$=\; \left(1 - \frac{1 + \mu + \frac{\mu^2}{2} + \ldots + \frac{\mu^{k-1}}{(k-1)!}}{e^{\mu}}\right)^n . \quad \square$$

For example, $n = 500$ on $A = 10^6$ m^2 with $r_0 = 90$ m yields $P(2\text{–con}) \approxeq (1 - e^{-\mu}(1+\mu))^n = 98\%$. In this paper, we say that probabilities larger than 0.95 are "high probabilities," and we denote such probabilities as p_{hi}. Furthermore, we say that $n \gg 1$ is fulfilled if $n \geq 1500\, r_0^2\pi/A$, and $r_0^2\pi \ll A$ is fulfilled if $r_0^2\pi \leq 0.08A$. With these values, the Poisson distribution used in (2) is a very good approximation for the binomial distribution. Note that μ denotes the expected number of neighbors of a node, i.e., the expected node degree $E(d)$. Further note that $\sum_{i=0}^{\infty} \frac{\mu^i}{i!} = e^{\mu}$ as $k \to \infty$, and, thus, $P(k\text{–con}) \to 0$.

A rearrangement of (2) for $k = 1$ and 2 gives us explicit formulas for the threshold ranges $r_0(P(\text{con}) = p_{hi})$ and $r_0(P(2\text{–con}) = p_{hi})$ that are necessary and sufficient to achieve a connected or 2–connected network with probability p_{hi}.

Corollary: If we want to be sure that a homogeneous wireless multihop network ($n \gg 1$, $\rho = n/A$) is connected with a probability of $P(\text{con})$, we can set the range of all nodes to

$$r_0 \approxeq \sqrt{\frac{\ln\left(1 - P(\text{con})^{1/n}\right)}{-\rho\pi}}\,, \tag{3}$$

where $P(\text{con})$ must be close to 1. Choosing r_0 above this threshold range increases $P(\text{con})$. Similarly, we must set

$$r_0 \approxeq \sqrt{\frac{W_{-1}\left((P(2\text{–con})^{1/n} - 1)\, e^{-1}\right) + 1}{-\rho\pi}}, \tag{4}$$

to achieve a 2–connected network with a probability of $P(2\text{–con})$. The function $W_{-1}(\cdot)$ denotes the real–valued, non–principal branch of the Lambert W function[1], as defined in [10]. Again, this equation is valid for high $P(2\text{–con})$. $\square$

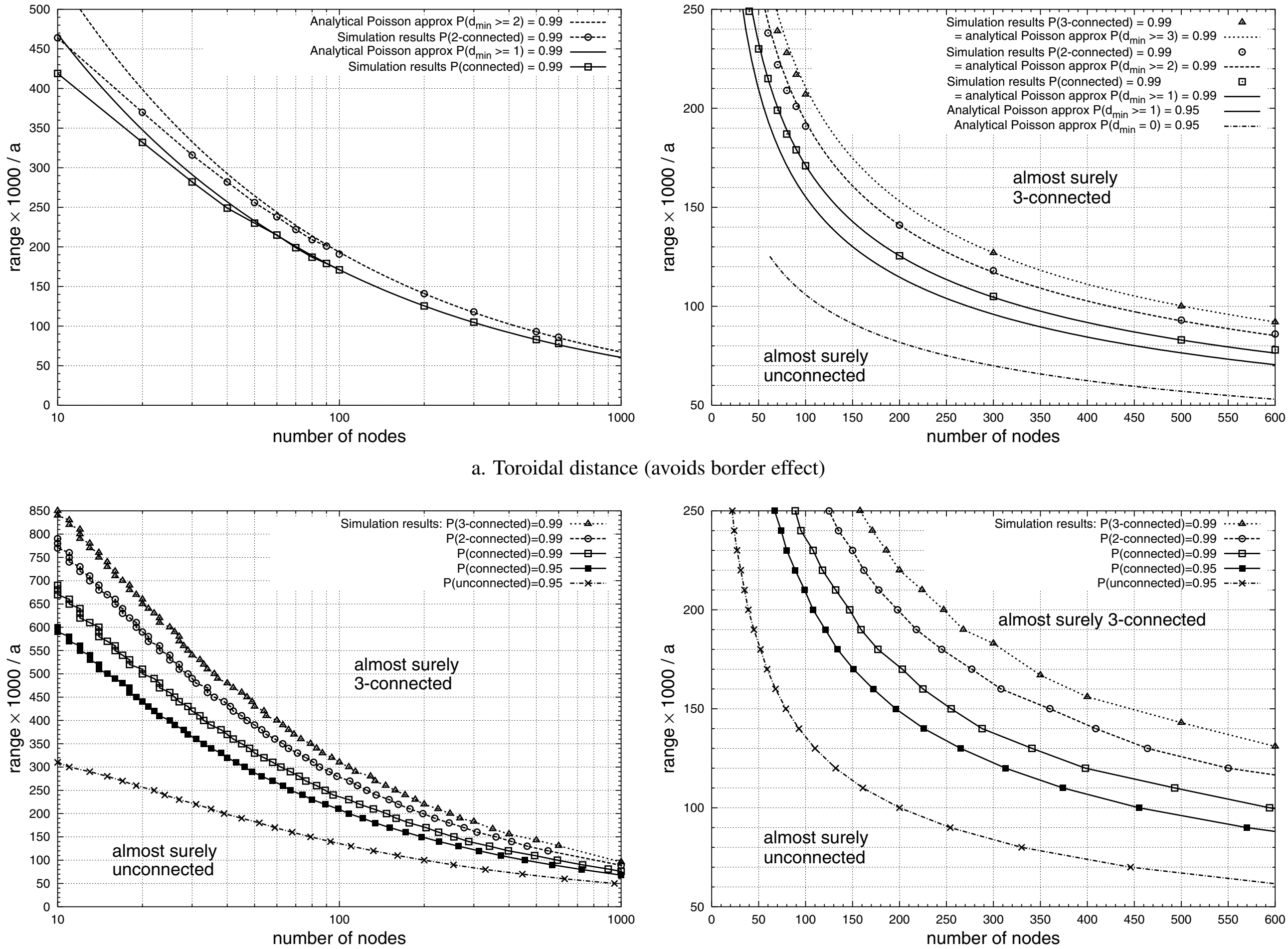
For illustration and validation of these results, we again consider a $1000 \times 1000\,\text{m}^2$ system area. Fig. 2a shows the analytical approximations and simulation results of $r_0(P(k\text{–con}) = 99\%)$ over n. To achieve with probability 99% a k–connected topology, we can choose any (r_0, n) pair that lies above the corresponding curve. The analytical values can be used for $n \geq 100$, which is required by the Poisson approximation. As shown on the y axis, the critical r_0 values can be scaled to any $a \times a$ area.

This presentation of critical (r_0, n) pairs is useful in practice for the design and simulation of wireless multihop networks. For example, a large–scale wireless sensor network should be distributed over a certain area, where the used sensor type can transmit a range r_0 in the given environment. We can now say how many sensors of this type are needed to obtain, almost surely, a k–connected network. Our results can also be employed in simulations with mobile nodes, if we use a mobility model that achieves a uniform spatial node distribution (see [11][6] for a discussion on this topic). For example, several simulation–based performance evaluations of routing protocols for ad hoc networks assume that the topology is connected during most of the simulation time (e.g., [12]). With our results, the simulation parameters can be set accordingly.

However, as already mentioned, these results are only applicable if border effects are not present or can be ignored in the regarded system area A. For example, no border effects occur, if the regarded area A is an inner subarea of a larger area A_+ (density ρ) and each borderline of A is at most r_0 away from the borderline of A_+. For example, one could be interested in the performance of a wireless multihop network on a university campus (area A). Nodes within the campus can use other nodes outside of the campus as relay nodes, but these outside nodes are not considered for the performance of the campus network. Another method to avoid border effects in simulations is to allow that nodes close to the border of A may have links to nodes at the opposite border of A via the borderline (toroidal distance).

If border effects are considered, nodes close to the border have a higher isolation probability, and therefore the overall connectivity decreases. A higher r_0 (or n) is needed to achieve

[1] The definition of the Lambert W function is that it satisfies $W(x)\, e^{W(x)} = x$. If x is a real number, two real values for $W(x)$ are possible for $-e^{-1} \leq x \leq 0$: the principal branch $W_0(x)$ with $W_0(x) \geq -1$, and a 2nd branch $W_{-1}(x)$ with $W_{-1}(x) \leq -1$. In our problem, $W_{-1}(x)$ yields the desired r_0, while $W_0(x)$ would result in a complex value. We used `LambertW(·)` in *Maple*.



a. Toroidal distance (avoids border effect)

b. Usual Euclidian distance (includes border effect)

Fig. 2. Required (r_0, n) pairs to achieve $P(k\text{–con}) \approx 1$ on an $a \times a$ system area. Simulations were made for $a = 1000$ m.

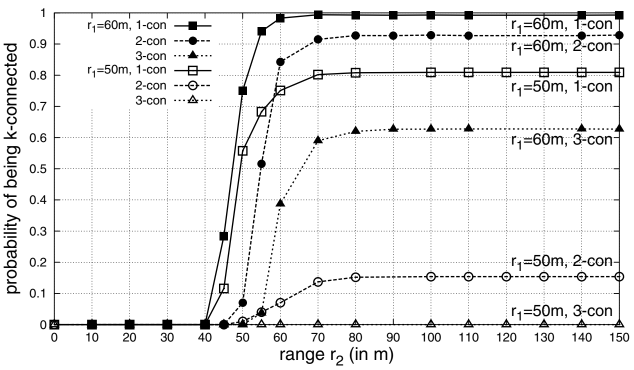
the same $P(k\text{–con})$. While in the toroidal model the connectivity just depends on the values $\rho = n/A$ and r_0, the size and shape of the area has significant influence on connectivity if we take border effects into account. In order to obtain the threshold ranges for a.s. k–connectivity in networks with border effects on a square area $a \times a$, we performed extensive simulations on $1000 \times 1000\ \text{m}^2$ for $P(k\text{–con}) = 99\%$. Fig. 2b shows the resulting (r_0, n) threshold pairs. For example, we need $n \approx 255$ nodes instead of $n \approx 130$ with $r_0 = 150$ m to make the network connected with probability 99%. Again, we can scale the results for r_0 to any square area. For example, on $800 \times 800\ \text{m}^2$, $n = 255$ nodes with $r_0 = 120$ m or, equivalently, $n \approx 165$ nodes with $r_0 = 150$ m are needed for $P(\text{con}) = 99\%$.

In both sets of simulation results (Fig. 2a and b), one point represents the sample average over 10000 random topologies. This average achieves the desired percentage of k–connected topologies (99%) within a tolerance of $\pm 0.5\%$. To obtain these results, we implemented a control loop: it first estimates the required r_0 for given n, simulates 1000 random topologies with these values, and then, based on the resulting estimate $\hat{P}(k\text{–con})$ increases or decreases r_0 until $\hat{P}(k\text{–con}) = 0.99 \pm 0.005$ is obtained. Finally, 10000 topologies are simulated with this r_0, and it is again checked whether $\hat{P}(k\text{–con}) = 0.99 \pm 0.005$.

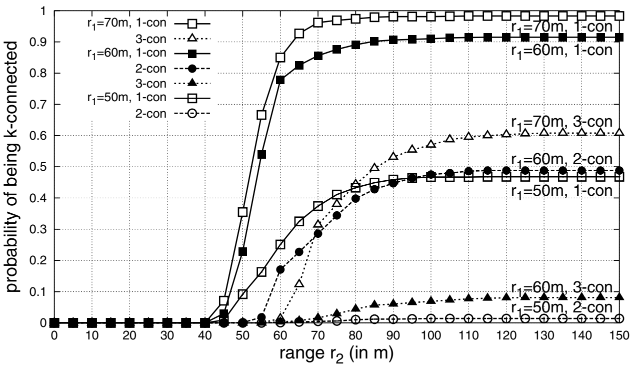
III. Inhomogeneous Range Assignment

Let us now go one step further and consider the question "which $P(\text{con})$ will be achieved, if the nodes do not have the same range?" To approach the solution to this problem, we first consider a scenario with two different ranges, i.e., n_1 nodes out of n nodes have range r_1, and the remaining $n_2 = n - n_1$ nodes have r_2. For example, two types of sensors are used for environmental monitoring in free space: type 1 is capable of transmitting up to a distance r_1, and the other achieves r_2.

Let us first investigate the case $n_1 = n_2 = 0.5\, n$. Again we place $n = 1000$ nodes on $1000 \times 1000\,\text{m}^2$. Fig. 3a shows $P(k\text{–con})$ for a fixed r_1 and varying r_2. Border effects are avoided. Regarding the curve for $r_1 = 50$ m and 1–connectivity, we make the following observations: For small r_2, $P(\text{con})$ shows a similar behavior as if we assigned $r_0 = r_2$ to all 1000 nodes. As in Fig. 1, $r_2 = r_1 = r_0 = 50$ m yields $P(\text{con}) = 0.55$. However, beyond this point, the curve levels off and finally achieves a saturation of $P(\text{con})_{sat} = 0.81$. Our interpretation is as follows: Increasing r_2 from 0 to 50 m helps both types of nodes to reduce their isolation probability

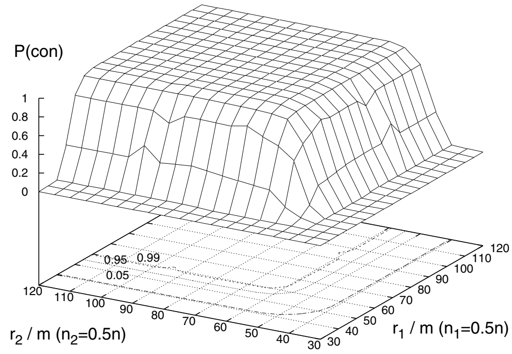


a. Toroidal distance (no border effects)

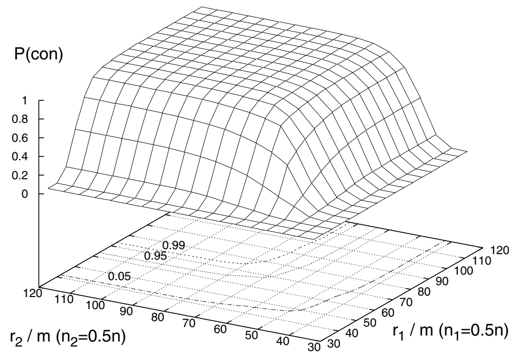


b. Usual Euclidian distance

Fig. 3. Connectivity for $n = 1000$ on 1000^2 m^2; r_1 fixed, $n_1 = n_2 = 0.5\,n$



a. Toroidal distance (no border effects)



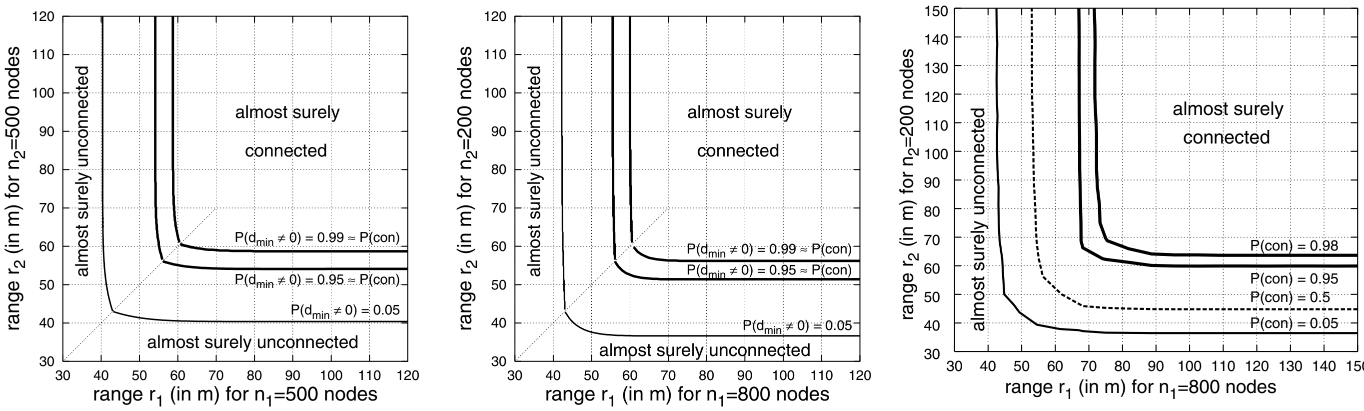
b. Usual Euclidian distance

Fig. 4. Connectivity for $n = 1000$ on $A = 1000^2$ m^2; $n_1 = n_2 = 0.5\,n$.

$P(d = 0)$. Increasing r_2 further ($r_2 > r_1$) only helps the nodes of type 2 to reduce $P(d = 0)$. The isolation probability for nodes of type 1 remains constant for $r_2 \geq r_1$, since, as mentioned above, we only consider bidirectional links. Finally, using nodes of type 2 with $r_2 > 70$ m yields almost no gain in 1–connectivity, since all of the $n_2 = 500$ nodes are a.s. non–isolated already for $r_2 = 70$ m. In the case with $r_1 = 60$ m, it is possible to achieve an a.s. connected topology, since in this case both node types can achieve a low isolation probability. The same qualitative behavior can be observed for $k = 2$ and 3 and the case with border effects (Fig. 3b). Fig. 4 shows $P(\text{con})$ for varying r_1 and r_2. We continue our conclusions: A necessary condition to achieve an a.s. connected topology with two node types of different range is that *both* ranges must be higher than a certain critical range. In our example, both r_1 and r_2 must be larger than 54 m without border effects, and 63 m with border effects, to enable $P(\text{con}) = 95\%$. We denote this critical range as the *required range.* If at least one of the ranges is smaller than the required range, it is impossible to achieve an a.s. connected network, even if the other range is very high.

Let us now drop the assumption $n_1 = n_2$ and consider any pair $n_1 < n$ and $n_2 = n - n_2$. Without loss of generality, we assume for a moment that $r_1 \geq r_2$. A node of type 1 is isolated if both of the following conditions are fulfilled: no other node of type 1 lies within distance r_1, and no other node of type 2 lies within distance r_2. If we ignore border effects and ensure $n_1, n_2 \gg 1$, the probability for this event is $P(\text{node with } r_1 \text{ isolated}) = e^{-\rho_1 r_1^2 \pi} \cdot e^{-\rho_2 r_2^2 \pi} = e^{-(\rho_1 r_1^2 + \rho_2 r_2^2)\,\pi}$, with $\rho_j = n_j / A$ ($j = 1, 2$). Note that $(\rho_1 r_1^2 + \rho_2 r_2^2)\,\pi$ is the expected degree of nodes of type 1. A node of type 2 is isolated if it has no other node (type 1 or 2) within distance r_2, i.e., $P(\text{node with } r_2 \text{ isolated}) = e^{-\rho_1 r_2^2 \pi} \cdot e^{-\rho_2 r_2^2 \pi} = e^{-\rho r_2^2 \pi}$. Let $P(d_{min}^{(j)} \neq 0)$ denote the probability that none of the n_j nodes of type j with r_j is isolated ($j = 1, 2$). We obtain $P(d_{min}^{(j)} \neq 0) = (1 - P(\text{node with } r_j \text{ isolated}))^{n_j}$ and $P(d_{min} \neq 0) = P(d_{min}^{(1)} \neq 0) \cdot P(d_{min}^{(2)} \neq 0)$. As in the homogeneous case, $P(\text{con}) \approxeq P(d_{min} \neq 0)$ can be applied for probabilities close to one.

Fig. 5 gives an example for $n_1 = n - n_2 = 0.8\,n$ and compares the result with the case $n_1 = n - n_2 = 0.5\,n$. Again, a necessary but not sufficient condition for $P(\text{con}) \geq p_{hi}$ is that both node types must outrange a certain critical range. This so–called required range is in general different for each node type and depends the density of the node type. Without border effects, if $r_1 \leq 55.5$ m or / and $r_2 \leq 51$ m, the network can never achieve $P(\text{con}) \geq 0.95$. If we request $P(\text{con}) \geq 0.99$, the necessary condition is that $r_1 \geq 60$ m and $r_2 \geq 56$ m are both fulfilled. If r_1 is so large that all nodes of type 1 are almost surely not isolated, i.e., $P(d_{min}^{(1)} \neq 0) \to 1$ and thus $P(d_{min} \neq 0) \to P(d_{min}^{(2)} \neq 0)$, the critical range of type 2 for $P(\text{con}) \geq p_{hi}$ is also a *sufficient* range for $P(\text{con}) \geq p_{hi}$ (see Fig. 5). If border effects are present, the required ranges are significantly higher. Note that the diagonal line $r_1 = r_2$ represents



a. $n_1 = 0.5n$, no border effects, analytical b. $n_1 = 0.8n$, no border effects, analytical c. $n_1 = 0.8n$, Euclidian dist., simulation results

Fig. 5. Connectivity for $n = 1000$ nodes with two different ranges on $A = 1000^2$ m^2

homogeneous range assignments (compare with Fig. 1). Let us generalize our results to the case with more than two ranges.

Theorem 2 (On the connectivity of ad hoc networks using nodes with different ranges): Given is a wireless multihop network with n uniformly distributed nodes. These n nodes consist of J different node types; i.e., there are n_j nodes of type j with range r_j, such that $n = \sum_{j=1}^{J} n_j$ for $j = 1 \ldots J$. We require $n_j \gg 1 \; \forall j$ and define the density $\rho_j = n_j/A$. If there are no border effects, we obtain

$$P(\text{node with } r_j \text{ isolated}) = \exp\Big(-\sum_{m=1}^{J} \rho_m \, r_e^2 \pi \Big) \qquad (5)$$

with the "effective range" $r_e = \min\{r_j, r_m\}$. Thus,

$$P(d_{min}^{(j)} \neq 0) = \Big(1 - P(\text{node with } r_j \text{ isolated}) \Big)^{n_j}, \qquad (6)$$

$$P(d_{min} \neq 0) = \prod_{j=1}^{J} P(d_{min}^{(j)} \neq 0). \qquad (7)$$

Finally, $P(\text{con}) \approxeq P(d_{min} \neq 0)$ for $P(d_{min} \neq 0)$ close to 1. □

If at least one of the factors $P(d_{min}^{(j)} \neq 0)$ is smaller than a value p, it is impossible to obtain $P(\text{con}) \geq p$. In other words, $P(\text{con}) \lesssim \min_j P(d_{min}^{(j)} \neq 0)$. Let us express this statement in terms of the required range of node type j, denoted as $r'_j(P(\text{con}) = p_{hi})$. The condition $r_j \geq r'_j(P(\text{con}) = p_{hi}) \; \forall j$ is a necessary but not sufficient condition to achieve a connected network with probability p_{hi}. To compute the required range of node type j, we set $r_m \to \infty$ for all other node types, i.e., for $m \in \{1, \ldots, J\} \setminus \{j\}$. From Equ. (6), we obtain

$$r'_j \approxeq \sqrt{\frac{\ln\left(1 - P(\text{con})^{1/n_j}\right)}{-\rho\pi}} \; . \qquad (8)$$

We conclude with an example. Let j_- denote the node type with the lowest and j_+ the type with the highest density. We observe from (8) that type j_- has the lowest and j_+ the highest required range. If we assign to each node the required range of the respective node type, i.e., $r_j = r'_j(P(\text{con}) = p_{hi}) \; \forall j$, we obtain $P(d_{min}^{(j_-)} \neq 0) = p_{hi}$ for the node type j_-. This non–isolation probability degrades for higher density nodes. If we set all ranges to r'_{j_+}, we obtain $P(d_{min}^{(j_+)} \neq 0) = p_{hi}$ for the node type j_+, and, as above, better non–isolation probabilities for lower density nodes.

IV. Conclusions

The results of this paper enable us to choose the parameters *transmission range* and *number of nodes* to achieve an a.s. k–connected wireless multihop network with uniformly distributed nodes. We considered scenarios with and without border effects and investigated inhomogeneous range assignments.

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